

PFAFFIAN EQUATIONS AND CONTIGUITY RELATIONS OF THE HYPERGEOMETRIC FUNCTION OF TYPE $(k+1, k+n+2)$ AND THEIR APPLICATIONS

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ABSTRACT. We study the structures of Pfaffian equations and contiguity relations of the hypergeometric function of type $(k+1, k+n+2)$ by using twisted cohomology groups and the intersection form on them. We apply our results to algebraic statistics; numerical evaluation of the normalizing constants of two way contingency tables with fixed marginal sums.

1. INTRODUCTION

We consider the hypergeometric integral of type $(k+1, k+n+2)$ defined as

$$F(\alpha; x) = \int_{\square} \prod_{i=1}^k t_i^{\alpha_i} \cdot \prod_{j=1}^n \left(1 + \sum_{i=1}^k x_{ij} t_i\right)^{\alpha_{k+j}} \\ \cdot \left(1 + \sum_{i=1}^k t_i\right)^{\alpha_{k+n+1}} \cdot \frac{dt_1 \wedge \cdots \wedge dt_k}{t_1 \cdots t_k},$$

where α_i 's are parameters, x_{ij} 's are $k \times n$ variables, and \square is a certain region. In this paper, we study Pfaffian equations and contiguity relations of the hypergeometric function $F(\alpha; x)$ by using the twisted cohomology groups and the intersection forms. In [M2] and [G], a Pfaffian equation and contiguity relations of Lauricella's F_D (the case of $k=1$) are studied in the same framework. This paper generalizes these results.

We regard Pfaffian equations and contiguity relations as matrix representations of some linear maps on twisted cohomology groups. To obtain the matrices providing the relations, we use the intersection form of the twisted cohomology group (Proposition 3.9 and Theorem 5.3). Our

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expressions have simple forms; each of the matrices for Pfaffian equations is determined by only an eigenvector with non-zero eigenvalue, and that for contiguity relations is decomposed into a product of intersection matrices and a diagonal one. An advantage of our method is that it systematically yields the relations from small initial data for a given basis of the twisted cohomology group without complicated calculations. Further, we give expressions of these linear maps by using the intersection forms (Theorems 3.12 and 5.5). They are independent of choice of bases of twisted cohomology groups.

Finally, we discuss an application of our results to algebraic statistics. We can express the normalizing constant of the hypergeometric distribution of the $r_1 \times r_2$ two way contingency tables with fixed marginal sums by the hypergeometric function of type $(r_1, r_1 + r_2)$ with integral parameters. By using contiguity relations, we give an algorithm (Algorithm 7.8) for evaluating values of the normalizing constant in the framework of holonomic gradient method [N3OST2]. Further, a Pfaffian equation gives the gradient matrix of the expectations, which is important to solve the conditional maximal likelihood estimate problem [TKT, §4]. We refer [Og] and [TKT] for statistical applications of hypergeometric functions.

Pfaffian equations and contiguity relations have been studied from several points of view. In [KM], Kita and the second author give an expression of the Gauss-Manin connection for some basis of the twisted cohomology group. In fact, it does not directly imply a Pfaffian equation; see §4. Aomoto studies the contiguity relations of the hypergeometric functions of type (k, n) by using twisted cohomology groups in [A]. This result is based on calculations in only the target space of \mathcal{U}_i in Proposition 5.2. On the other hand, by considering both of its domain and target spaces, we can clarify a structure of contiguity relations. Sasaki studies them in the framework of the Aomoto-Gel'fand system on the Grassmannian manifold in [S]. However, it only gives contiguity relations on coordinates of the Grassmannian manifold in general case. Though Takayama gives an algorithmic method that uses Gröbner bases to derive the contiguity relations in [T], this method requires huge computer resources. Recently in [OhT], Ohara and Takayama give a numerical method to derive Pfaffian equations and contiguity relations of A -hypergeometric systems, one of which $F(\alpha; x)$ satisfies, to evaluate the normalizing constant of A -hypergeometric distributions, but it is still difficult to get Pfaffian equations and contiguity relations unless k and n are small enough.

For statistical applications, the method given in [OhT] is applicable to evaluation not only for two way contingency tables but also for other

cases, while our algorithm is much faster and can solve larger problems than theirs for the two way contingency tables.

2. PRELIMINARIES

Let $Z = (z_{ij})$ be a square matrix arranged n^2 variables z_{ij} ($1 \leq i, j \leq n$).

Fact 2.1. *The logarithmic derivative of the determinant $|Z|$ of Z is*

$$d \log |Z| = \sum_{1 \leq i, j \leq n} \frac{(-1)^{i+j} |Z_{ij}| dz_{ij}}{|Z|},$$

where Z_{ij} is the square matrix of size $n-1$ removing the i -th row and the j -th column from Z , and $|Z_{ij}|$ is its determinant.

Proof. By the cofactor expansion

$$|Z| = \sum_{p=1}^n (-1)^{p+i} z_{ip} |Z_{ip}|,$$

with respect to the i -th row of Z , we have

$$d|Z| = \sum_{p=1}^n \left((-1)^{p+i} |Z_{ip}| dz_{ip} + (-1)^{p+i} z_{ip} d|Z_{ip}| \right).$$

Since the minor $|Z_{ip}|$ does not have the variable z_{ij} , the coefficient function of dz_{ij} in $d|Z|$ is $(-1)^{i+j} |Z_{ij}|$. This property together with the symmetry yields this fact. \square

Let x_{ij} ($1 \leq i \leq k$, $1 \leq j \leq n$) be $k \times n$ variables and $x = (x_{ij})$ be the matrix arranging them. We set a $(k+1) \times (k+n+2)$ matrix

$$\begin{aligned} \tilde{x} &= (\tilde{x}_{ij})_{\substack{0 \leq i \leq k \\ 0 \leq j \leq k+n+1}} = \begin{pmatrix} 1 & \mathbf{0}_k & \mathbf{1}_n & 1 \\ {}^t\mathbf{0}_k & I_k & x & {}^t\mathbf{1}_k \end{pmatrix} \\ &= \begin{matrix} & 0 & 1 & \cdots & k & k+1 & \cdots & k+n & k+n+1 \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ k \end{matrix} & \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ 0 & 1 & & 0 & x_{11} & \cdots & x_{1n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & x_{k1} & \cdots & x_{kn} & 1 \end{pmatrix} \end{matrix}, \end{aligned}$$

where $\mathbf{0}_k = (0, \dots, 0) \in \mathbb{Z}^k$, $\mathbf{1}_n = (1, \dots, 1) \in \mathbb{Z}^n$, and I_k is the unit matrix of size k .

Let \mathcal{J} be the set of subsets of $\{0, 1, 2, \dots, k+n, k+n+1\}$ with cardinality $k+1$. Any element in \mathcal{J} is expressed as

$$J = \{j_0, j_1, \dots, j_k\}, \quad 0 \leq j_0 < j_1 < \cdots < j_k \leq k+n+1.$$

We set

$$\tilde{x}\langle J \rangle = \begin{pmatrix} \tilde{x}_{0,j_0} & \tilde{x}_{0,j_1} & \cdots & \tilde{x}_{0,j_k} \\ \tilde{x}_{1,j_0} & \tilde{x}_{1,j_1} & \cdots & \tilde{x}_{1,j_k} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{k,j_0} & \tilde{x}_{k,j_1} & \cdots & \tilde{x}_{k,j_k} \end{pmatrix},$$

which is the sub-matrix of \tilde{x} consisting of the j_0 -th, j_1 -th, \dots , j_k -th columns. We define a subset of \mathcal{J} by

$$\mathcal{J}^\circ = \{J \in \mathcal{J} \mid d_x |\tilde{x}\langle J \rangle| \neq 0\},$$

where d_x is the exterior derivative with respect to $x_{11}, x_{12}, \dots, x_{kn}$.

Lemma 2.2. *Any element J of \mathcal{J}° does not include an index i with $1 \leq i \leq k$ and includes an index $k+j$ with $1 \leq j \leq n$.*

Proof. If $J \in \mathcal{J}$ includes $1, \dots, k$ then $|\tilde{x}\langle J \rangle| = \pm 1$. If J includes none of $k+1, \dots, k+n$, then there is no variable x_{ij} in $\tilde{x}\langle J \rangle$. \square

We count the cardinality of the set \mathcal{J}° . It is easy to that $\#(\mathcal{J}) = \binom{k+n+2}{k+1}$. If we choose $J = \{j_0, j_1, \dots, j_k\} \in \mathcal{J}$ so that

$$j_0 = 1, j_1 = 2, \dots, j_{k-1} = k,$$

then $\tilde{x}\langle J \rangle$ becomes a constant for any j_k . There are $n+2$ ways to choose j_k . If we choose

$$j_0 = 0, j_k = n+k+1,$$

then $\tilde{x}\langle J \rangle$ becomes a constant for $\{j_1, \dots, j_{k-1}\} \subset \{1, \dots, k\}$. There are k ways to choose j_1, \dots, j_{k-1} . Thus we have the following lemma.

Lemma 2.3. *The cardinality of \mathcal{J}° is*

$$\binom{k+n+2}{k+1} - (k+n+2).$$

Let L_j ($0 \leq j \leq k+n+1$) be linear forms of t_0, t_1, \dots, t_k defined by

$$(t_0, t_1, \dots, t_k)\tilde{x} = (L_0, L_1, \dots, L_k, L_{k+1}, \dots, L_{k+n}, L_{k+n+1}).$$

Namely,

$$L_j = t_j \quad (0 \leq j \leq k), \quad L_{k+j} = t_0 + t_1 x_{1j} + \cdots + t_k x_{kj} \quad (1 \leq j \leq n),$$

$$L_{k+n+1} = t_0 + t_1 + \cdots + t_k.$$

Lemma 2.4. *Let $J = \{j_0, \dots, j_k\}$ be an element of \mathcal{J} . The linear form $t_i = L_i$ ($0 \leq i \leq k$) can be expressed in terms of L_{j_0}, \dots, L_{j_k} as*

$$t_i = \frac{1}{|\tilde{x}\langle J \rangle|} \sum_{p=0}^k |\tilde{x}\langle j_p J_i \rangle| L_{j_p},$$

where $j_p J_i = (J - \{j_p\}) \cup \{i\} = \{j_0, \dots, j_{p-1}, i, j_{p+1}, \dots, j_k\}$.

Proof. Since

$$(t_0, t_1, \dots, t_k) \tilde{x}\langle J \rangle = (L_{j_0}, L_{j_1} \dots, L_{j_k}),$$

t_i 's are expressed as

$$(t_0, t_1, \dots, t_k) = (L_{j_0}, L_{j_1} \dots, L_{j_k}) \tilde{x}\langle J \rangle^{-1}.$$

We have only to write down the i -th column of the cofactor matrix of $\tilde{x}\langle J \rangle$. \square

We regard (t_0, t_1, \dots, t_k) as projective coordinates of \mathbb{P}^k and (t_1, \dots, t_k) as affine coordinates with setting $t_0 = 1$. For $J = \{j_0, j_1, \dots, j_k\} \in \mathcal{J}$ we set

$$\varphi\langle J \rangle = d_t \log(L_{j_1}/L_{j_0}) \wedge d_t \log(L_{j_2}/L_{j_0}) \wedge \dots \wedge d_t \log(L_{j_k}/L_{j_0}),$$

where d_t is the exterior derivative with respect to t_1, \dots, t_k .

Fact 2.5. *We have*

$$\varphi\langle J \rangle = \frac{|\tilde{x}\langle J \rangle| dt}{\prod_{p=0}^k L_{j_p}}, \quad dt = dt_1 \wedge \dots \wedge dt_k.$$

Proof. We use the following identity:

$$(2.1) \quad \varphi\langle J \rangle = \sum_{p=0}^k (-1)^p L_{j_p} \frac{\bigwedge_{0 \leq q \leq p}^{q \neq p} d_t L_{j_q}}{\prod_{q=0}^k L_{j_q}}.$$

Consider the cofactor expansion of the 0-th row of the sub-matrix $\tilde{x}\langle J \rangle$ of \tilde{x} . \square

3. GAUSS-MANIN CONNECTIONS

Let $\alpha_0, \alpha_1, \dots, \alpha_{k+n}, \alpha_{k+n+1}$ be parameters in $\mathbb{C} - \mathbb{Z}$ satisfying

$$(3.1) \quad \sum_{j=0}^{k+n+1} \alpha_j = 0.$$

We set $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{k+n}, \alpha_{k+n+1})$. We often regard α_i 's as indeterminants. For an element $f(\alpha)$ of the rational function field $\mathbb{C}(\alpha) = \mathbb{C}(\alpha_0, \dots, \alpha_{k+n+1})$, we put $f(\alpha)^\vee = f(-\alpha)$. For a matrix A with entries in $\mathbb{C}(\alpha)$, we denote by A^\vee the matrix operated $^\vee$ on each entry of A .

We define sets X and \mathfrak{X} as

$$\begin{aligned} X &= \{x \in M(k, n; \mathbb{C}) \mid |\tilde{x}\langle J \rangle| \neq 0 \text{ for any } J\}, \\ \mathfrak{X} &= \{(t, x) \in \mathbb{C}^k \times X \mid \prod_{j=0}^{k+n+1} L_j \neq 0\}. \end{aligned}$$

We set 1-forms ω and ω_x as

$$\begin{aligned}
\omega &= \sum_{j=1}^{k+n+1} \alpha_j d_t \log L_j \\
&= \alpha_1 \frac{dt_1}{t_1} + \cdots + \alpha_k \frac{dt_k}{t_k} + \alpha_{k+1} \frac{x_{11} dt_1 + \cdots + x_{k1} dt_k}{1 + t_1 x_{11} + \cdots + t_k x_{k1}} + \cdots \\
&\quad + \alpha_{k+n} \frac{x_{1n} dt_1 + \cdots + x_{kn} dt_k}{1 + t_1 x_{1n} + \cdots + t_k x_{kn}} + \alpha_{k+n+1} \frac{dt_1 + \cdots + dt_k}{1 + t_1 + \cdots + t_k}, \\
\omega_x &= \sum_{j=1}^n \alpha_{k+j} d_x \log L_{k+j} = \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} \frac{\alpha_{k+j} t_i dx_{ij}}{L_{k+j}} \\
&= \frac{\alpha_{k+1} t_1 dx_{11}}{1 + t_1 x_{11} + \cdots + t_k x_{k1}} + \cdots + \frac{\alpha_{k+1} t_k dx_{k1}}{1 + t_1 x_{11} + \cdots + t_k x_{k1}} \\
&\quad + \cdots + \frac{\alpha_{k+j} t_i dx_{ij}}{1 + t_1 x_{1j} + \cdots + t_k x_{kj}} + \cdots \\
&\quad + \frac{\alpha_{k+n} t_1 dx_{1n}}{1 + t_1 x_{1n} + \cdots + t_k x_{kn}} + \cdots + \frac{\alpha_{k+n} t_k dx_{kn}}{1 + t_1 x_{1n} + \cdots + t_k x_{kn}}.
\end{aligned}$$

We define operators as

$$\nabla^\alpha = d_t + \omega \wedge, \quad \nabla_x^\alpha = d_x + \omega_x \wedge, \quad \nabla_{ij}^\alpha = \frac{\partial}{\partial x_{ij}} + \frac{\alpha_{k+j} t_i}{L_{k+j}}.$$

Note that

$$(3.2) \quad \nabla_x^\alpha = \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} dx_{ij} \wedge \nabla_{ij}^\alpha.$$

For a fixed $x \in X$, we have twisted cohomology groups

$$\begin{aligned}
H^k(\Omega^\bullet(T_x), \nabla^\alpha) &= \Omega^k(T_x) / \nabla^\alpha(\Omega^{k-1}(T_x)), \\
H^k(\Omega^\bullet(T_x), \nabla^{-\alpha}) &= \Omega^k(T_x) / \nabla^{-\alpha}(\Omega^{k-1}(T_x)),
\end{aligned}$$

where T_x is the preimage of x under the projection $\mathfrak{X} \ni (t, x) \mapsto x \in X$, and $\Omega^l(T_x)$ is the vector space of rational l -forms on \mathbb{P}^k with poles only along $\mathbb{P}^k - T_x$. Here, we identify T_x with an open subset of \mathbb{P}^k .

Fact 3.1 ([AK]). *The twisted cohomology groups $H^k(\Omega^\bullet(T_x), \nabla^{\pm\alpha})$ are of rank $\binom{k+n}{k}$.*

There is the intersection pairing \mathcal{I} between $H^k(\Omega^\bullet(T_x), \nabla^\alpha)$ and $H^k(\Omega^\bullet(T_x), \nabla^{-\alpha})$.

Fact 3.2 ([M1]). For $J = \{j_0, \dots, j_k\}$ and $J' = \{j'_0, \dots, j'_k\}$, we have

$$\mathcal{I}(\varphi\langle J \rangle, \varphi\langle J' \rangle) = \begin{cases} (2\pi\sqrt{-1})^k \cdot \frac{\sum_{j \in J} \alpha_j}{\prod_{j \in J} \alpha_j} & \text{if } J = J', \\ (2\pi\sqrt{-1})^k \cdot \frac{(-1)^{p+q}}{\prod_{j \in J \cap J'} \alpha_j} & \text{if } \#(J \cap J') = k, \\ 0 & \text{otherwise,} \end{cases}$$

where we assume that $J - \{j_p\} = J' - \{j'_q\}$ in the case of $\#(J \cap J') = k$.

Note that if we regard α_i 's as indeterminants, we have $\mathcal{I}(\varphi\langle J \rangle, \varphi\langle J' \rangle)^\vee = (-1)^k \cdot \mathcal{I}(\varphi\langle J \rangle, \varphi\langle J' \rangle)$.

Proposition 3.3. Let p and q be different two elements of the set $\{0, 1, \dots, k+n+1\}$. We set

$$\begin{aligned} {}_q\mathcal{J}_p &= \{J \in \mathcal{J} \mid q \notin J, p \in J\}, \\ {}_p\mathcal{J}_q &= \{J \in \mathcal{J} \mid p \notin J, q \in J\}. \end{aligned}$$

Then $\{\varphi\langle J \rangle\}_{J \in {}_p\mathcal{J}_q}$ and $\{\varphi\langle J' \rangle\}_{J' \in {}_q\mathcal{J}_p}$ are bases of $H^k(\Omega^\bullet(T_x), \nabla^{\pm\alpha})$.

Proof. Set $J = \{p, j_1, \dots, j_k\} \in {}_q\mathcal{J}_p$ and $J' = \{j_1, \dots, j_k, q\} \in {}_p\mathcal{J}_q$. Note that there are $\binom{k+n}{k}$ ways to get such J and J' for fixed p and q . We align $\varphi\langle J \rangle$'s and $\varphi\langle J' \rangle$'s by the lexicographic order of (j_1, \dots, j_k) . Then the intersection matrix for $\varphi\langle J \rangle$ and $\varphi\langle J' \rangle$ becomes diagonal matrix with diagonal entries

$$\frac{(-2\pi\sqrt{-1})^k}{\alpha_{j_1} \cdots \alpha_{j_k}}.$$

Thus they are bases of $H^k(\Omega^\bullet(T_x), \nabla^{\pm\alpha})$. \square

We define vector bundles of rank $r = \binom{k+n}{k}$ over X with fibers $H^k(\Omega^\bullet(T_x), \nabla^\alpha)$ and $H^k(\Omega^\bullet(T_x), \nabla^{-\alpha})$ by

$$\mathcal{H}^\alpha = \bigcup_{x \in X} H^k(\Omega^\bullet(T_x), \nabla^\alpha), \quad \mathcal{H}^{-\alpha} = \bigcup_{x \in X} H^k(\Omega^\bullet(T_x), \nabla^{-\alpha}),$$

respectively. We can regard $\{\varphi\langle J \rangle\}_{J \in {}_p\mathcal{J}_q}$ and $\{\varphi\langle J' \rangle\}_{J' \in {}_q\mathcal{J}_p}$ as global frames of these vector bundles. The operators ∇_x^α and $\nabla_x^{-\alpha}$ are regarded as connections on \mathcal{H}^α and $\mathcal{H}^{-\alpha}$, i.e., they are $\mathbb{C}(\alpha)$ -linear maps

$$\nabla_x^{\pm\alpha} : \Gamma(\mathcal{H}^{\pm\alpha}) \rightarrow \Gamma(\Omega^1(X) \otimes \mathcal{H}^{\pm\alpha})$$

satisfying

$$\nabla_x^{\pm\alpha}(f\varphi) = d_x f \otimes \varphi + f \nabla_x^{\pm\alpha}(\varphi),$$

for $f \in \Omega^0(X)$ and $\varphi \in \Gamma(\mathcal{H}^{\pm\alpha})$, where $\Omega^l(X)$ is a space of rational l -forms with poles only along the complement of X , $\Gamma(\mathcal{V})$ denotes

the $\Omega^0(X)$ -module of sections of \mathcal{V} . They are called the Gauss-Manin connections on the vector bundles $\mathcal{H}^{\pm\alpha}$. Note that

$$(3.3) \quad \frac{\partial}{\partial x_{ij}} \int_{\square} \left(\prod_{j=1}^{n+k+1} L_j^{\alpha_j} \right) \varphi = \int_{\square} \left(\prod_{j=1}^{n+k+1} L_j^{\alpha_j} \right) \nabla_{ij}^{\alpha}(\varphi),$$

$$(3.4) \quad d_x \int_{\square} \left(\prod_{j=1}^{n+k+1} L_j^{\alpha_j} \right) \varphi = \int_{\square} \left(\prod_{j=1}^{n+k+1} L_j^{\alpha_j} \right) \nabla_x^{\alpha}(\varphi),$$

for $\varphi \in \Gamma(\mathcal{H}^{\alpha})$, where \square is a twisted cycle associated with $\prod_{j=1}^{n+k+1} L_j^{\alpha_j}$ (refer to [AK, §3.2] for its definition). These mean that the partial differential operators $\frac{\partial}{\partial x_{ij}}$ and the exterior derivative d_x are translated into the operators ∇_{ij}^{α} and the connection ∇_x^{α} through the integration with respect to the kernel function $\prod_{j=1}^{n+k+1} L_j^{\alpha_j}$.

The intersection form \mathcal{I} is extended to a pairing between $\Gamma(\mathcal{H}^{\alpha})$ and $\Gamma(\mathcal{H}^{-\alpha})$. We can regard $\mathcal{H}^{-\alpha}$ as the dual of \mathcal{H}^{α} by the intersection form \mathcal{I} , since \mathcal{I} is a perfect pairing. By the compatibility of the connections and the intersection form, we have the following.

Proposition 3.4. *The intersection form \mathcal{I} satisfies*

$$d_x \mathcal{I}(\varphi, \varphi') = \mathcal{I}(\nabla_x^{\alpha}(\varphi), \varphi') + \mathcal{I}(\varphi, \nabla_x^{-\alpha}(\varphi')),$$

for $\varphi \in \Gamma(\mathcal{H}^{\alpha})$ and $\varphi' \in \Gamma(\mathcal{H}^{-\alpha})$.

Let $\Gamma_0(\mathcal{H}^{\alpha})$ (resp. $\Gamma_0(\mathcal{H}^{-\alpha})$) be the vector space in $\Gamma(\mathcal{H}^{\alpha})$ (resp. $\Gamma(\mathcal{H}^{-\alpha})$) spanned by $\varphi\langle J \rangle$'s ($J \in \mathcal{J}$) over the field $\mathbb{C}(\alpha)$. Then any elements $\varphi \in \Gamma_0(\mathcal{H}^{\alpha})$ and $\varphi' \in \Gamma_0(\mathcal{H}^{-\alpha})$ satisfy

$$(3.5) \quad d_x \mathcal{I}(\varphi, \varphi') = 0$$

by Fact 3.2.

We put $\dot{\mathcal{J}} = {}_{k+n+1}\mathcal{J}_0$, and align its elements lexicographically. We denote $\dot{\mathcal{J}} = \{J^1, J^2, \dots, J^r\}$, $r = \binom{k+n}{k}$, and define a column vector $\Phi = {}^t(\varphi\langle J^1 \rangle, \varphi\langle J^2 \rangle, \dots, \varphi\langle J^r \rangle)$. We compute the connection matrix $\Psi(\alpha; x)$ of ∇_x^{α} with respect to the frame $\{\varphi\langle J \rangle\}_{J \in \dot{\mathcal{J}}}$; i.e., it satisfies

$$(3.6) \quad \nabla_x^{\alpha} \Phi = \Psi(\alpha; x) \cdot \Phi.$$

We remark that $\Psi(\alpha; x)$ is a square matrix of size $\binom{k+n}{k}$ with entries in $\Omega^1(X)$. We also express ∇_x^{α} by the intersection form \mathcal{I} without taking a frame. By the decomposition (3.2) of ∇_x^{α} , we have

$$\Psi(\alpha; x) = \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} \Psi_{ij}(\alpha; x) dx_{ij},$$

where the matrix $\Psi_{ij}(\alpha; x)$ is obtained by the action of operator ∇_{ij}^α on the frame $\{\varphi\langle J \rangle\}_{J \in \mathcal{J}}$. We study the operator ∇_{ij}^α .

Lemma 3.5. *Let $J = \{j_0, \dots, j_k\}$ be an element of \mathcal{J} . Suppose that*

$$1 \leq i \leq k, \quad 1 \leq j \leq n, \quad k+j \notin J.$$

Then

$$\nabla_{ij}^\alpha(\varphi\langle J \rangle) = \alpha_{k+j} \sum_{p=0}^k \frac{|\tilde{x}\langle_{j_p} J_i \rangle|}{|\tilde{x}\langle_{j_p} J_{k+j} \rangle|} \varphi\langle_{j_p} J_{k+j} \rangle,$$

where

$$\begin{aligned} j_p J_i &= \{j_0, \dots, j_{p-1}, i, j_{p+1}, \dots, j_k\}, \\ j_p J_{k+j} &= \{j_0, \dots, j_{p-1}, k+j, j_{p+1}, \dots, j_k\}. \end{aligned}$$

Proof. Since $(\partial/\partial x_{ij})\varphi\langle J \rangle = 0$, we have

$$\begin{aligned} \nabla_{ij}^\alpha(\varphi\langle J \rangle) &= \frac{\alpha_{k+j} t_i}{L_{k+j}} \varphi\langle J \rangle \\ &= \frac{\alpha_{k+j}}{|\tilde{x}\langle J \rangle|} \left(\sum_{p=0}^k |\tilde{x}\langle_{j_p} J_i \rangle| L_{j_p} \right) \frac{|\tilde{x}\langle J \rangle| dt}{L_{k+j} L_{j_0} L_{j_1} \cdots L_{j_k}} \\ &= \alpha_{k+j} \sum_{p=0}^k \frac{|\tilde{x}\langle_{j_p} J_i \rangle|}{|\tilde{x}\langle_{j_p} J_{k+j} \rangle|} \varphi\langle_{j_p} J_{k+j} \rangle. \end{aligned}$$

Here we use Lemma 2.4 for the expression of t_i . \square

For a fixed i and j , we take a basis $\varphi\langle J \rangle$'s of $H^k(\Omega^\bullet(T_x), \nabla^\alpha)$, where J runs over the subset ${}_{k+j}\mathcal{J}_i = \{J \in \mathcal{J} \mid k+j \notin J, i \in J\}$ of \mathcal{J} with cardinality $\binom{k+n}{k}$. Under this condition, we have

$$\frac{|\tilde{x}\langle_{j_p} J_i \rangle|}{|\tilde{x}\langle_{j_p} J_{k+j} \rangle|} dx_{ij} = (-1)^{p+i} \frac{\partial}{\partial x_{ij}} \log |\tilde{x}\langle_{j_p} J_{k+j} \rangle| dx_{ij},$$

by Fact 2.1 and

$$\tilde{x}\langle_{j_p} J_i \rangle = i \begin{pmatrix} 0 & \cdots & p-1 & p & p+1 & \cdots & k \\ \tilde{x}_{0,j_0} & \cdots & \tilde{x}_{0,j_{p-1}} & 0 & \tilde{x}_{0,j_{p+1}} & \cdots & \tilde{x}_{0,j_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \tilde{x}_{i,j_0} & \cdots & \tilde{x}_{i,j_{p-1}} & 1 & \tilde{x}_{i,j_{p+1}} & \cdots & \tilde{x}_{i,j_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ k & \tilde{x}_{k,j_0} & \cdots & \tilde{x}_{k,j_{p-1}} & 0 & \tilde{x}_{k,j_{p+1}} & \cdots & \tilde{x}_{k,j_k} \end{pmatrix},$$

$$\tilde{x}_{\langle j_p J_{k+j} \rangle} = i \begin{pmatrix} 0 & \cdots & p-1 & p & p+1 & \cdots & k \\ \tilde{x}_{0,j_0} & \cdots & \tilde{x}_{0,j_{p-1}} & 1 & \tilde{x}_{0,j_{p+1}} & \cdots & \tilde{x}_{0,j_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{x}_{i,j_0} & \cdots & \tilde{x}_{i,j_{p-1}} & x_{i,j} & \tilde{x}_{i,j_{p+1}} & \cdots & \tilde{x}_{i,j_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{x}_{k,j_0} & \cdots & \tilde{x}_{k,j_{p-1}} & x_{k,j} & \tilde{x}_{k,j_{p+1}} & \cdots & \tilde{x}_{k,j_k} \end{pmatrix}.$$

Note that the basis change transformation matrix from $\{\varphi\langle J \rangle\}_{J \in k+j\mathcal{J}_i}$ to $\{\varphi\langle J \rangle\}_{J \in \mathcal{J}}$ is independent of x_{ij} . Thus the coefficient $\Psi_{ij}(\alpha; x)$ of dx_{ij} in the connection matrix $\Psi(\alpha; x)$ can be expressed as

$$(3.7) \quad \Psi_{ij}(\alpha; x) = \sum_{J \in \mathcal{J}^\circ, J \ni k+j} M_J^{ij}(\alpha) \frac{\partial}{\partial x_{ij}} \log |\tilde{x}\langle J \rangle|,$$

where $M_J^{ij}(\alpha)$ are square matrices of size $r = \binom{k+n}{k}$ which are independent of any entries of x .

Lemma 3.6. *Suppose that $\alpha_J = \alpha_{j_0} + \cdots + \alpha_{j_k} \neq 0$ for $J = \{j_0, \dots, j_k\} \in \mathcal{J}^\circ$ with $i \notin J$, $k+j \in J$.*

- (1) *The matrices $M_J^{ij}(\alpha)$ are independent of i and j .*
- (2) *The eigenvalues of the matrix $M_J^{ij}(\alpha)$ are α_J and 0.*
- (3) *Its eigenspace of eigenvalue α_J is 1-dimensional, and that of eigenvalue 0 is $(r-1)$ -dimensional.*

Proof. We may assume $j_0 = k+j$. We set

$${}_{j_0}\mathcal{J}_i'' = (\{J\} \cup {}_{j_0}\mathcal{J}_i) - \{j_0 J_i\},$$

and assume that its first entry is J .

We take a point $\dot{x} \in X$ so that L_j ($j \in J$) form a real small simplex. Let $U_{\dot{x}}$ be a small open set in $\mathbb{C}^{k \times n}$ including \dot{x} and points with $|\tilde{x}\langle J \rangle| = 0$. For any $J'' \in {}_{j_0}\mathcal{J}_i''$, we make a twisted cycle $\Delta\langle J'' \rangle$ by using L_j ($j \in J''$). By computing the intersection numbers of $\Delta\langle J'' \rangle$, we can show that they form a basis of a twisted homology group. We construct a period matrix

$$\Pi^\alpha(x) = \left(\int_{\Delta\langle J'' \rangle} \left(\prod_{j=1}^{k+n+1} L_j^{\alpha_j} \right) \varphi\langle J' \rangle \right)_{J' \in \mathcal{J}, J'' \in {}_{j_0}\mathcal{J}_i''}$$

on $U_{\dot{x}} \cap X$. By the perfectness of the pairing between the twisted homology and cohomology groups, it is invertible. When x turns around the divisor $|\tilde{x}\langle J \rangle| = 0$, the argument of each L_i on $\Delta\langle J'' \rangle$ is almost unchanged for $J'' \in {}_{j_0}\mathcal{J}_i'' - \{J\}$. Moreover, the integrals over $\Delta\langle J'' \rangle$ ($J'' \in {}_{j_0}\mathcal{J}_i'' - \{J\}$) are valid on the divisor $|\tilde{x}\langle J \rangle| = 0$ in $U_{\dot{x}}$. Thus the entries $\Pi^\alpha(x)$ except in the first column are single-valued and holomorphic

on $U_{\tilde{x}}$. We consider the behavior of the first column of $\Pi^\alpha(x)$. To compute the integrals, we use the coordinate change such that L_j ($j \in J$) are expressed as $t_1, \dots, t_k, 1 + t_1 + \dots + t_k$. This coordinate change is equivalent to the left multiplication \tilde{x}' of $\tilde{x}\langle 1, \dots, k, k+n+1 \rangle \tilde{x}\langle J \rangle^{-1}$ to \tilde{x} . Let L'_j be linear forms corresponding to the matrix \tilde{x}' . We have

$$\int_{\Delta\langle J \rangle} \left(\prod_{j=1}^{k+n+1} L_j^{\alpha_j} \right) \varphi\langle J' \rangle = \int_{\Delta'} \left(\prod_{j=1}^{k+n+1} L'_j{}^{\alpha_j} \right) \varphi'\langle J' \rangle,$$

where $\varphi'\langle J' \rangle$ is naturally defined by \tilde{x}' and Δ' is the regularization of a standard simplex

$$(3.8) \quad \Delta = \{(t_1, \dots, t_k) \in \mathbb{R}^k \mid t_1, \dots, t_k < 0, t_1 + \dots + t_k > -1\}$$

with respect to $\prod_{j=1}^{k+n+1} L'_j{}^{\alpha_j}$. Here note that every linear form L'_q ($q \notin J$) has the factor $|\tilde{x}\langle J \rangle|^{-1}$. By taking out this factor from this integral, we see that each entry in the first column of $\Pi^\alpha(x)$ is the product of

$$\prod_{q \notin J} |\tilde{x}\langle J \rangle|^{-\alpha_q} = |\tilde{x}\langle J \rangle|^{\alpha_J}$$

and a single-valued holomorphic function on $U_{\tilde{x}}$. Here note that we use the assumption (3.1). Hence we have a local expression

$$\Pi^\alpha(x) = \Pi_J^\alpha(x) D_J^\alpha(x), \quad D_J^\alpha(x) = \text{diag}(|\tilde{x}\langle J \rangle|^{\alpha_J}, 1, \dots, 1)$$

around \tilde{x} , where $\Pi_J^\alpha(x)$ is a single-valued holomorphic matrix function on $U_{\tilde{x}}$, and $\text{diag}(c_1, \dots, c_r)$ denotes the diagonal matrix with diagonal entries c_1, \dots, c_r . By operating d_x on the both sides of the above and using the equalities (3.4) and (3.6), we have

$$\begin{aligned} d_x \Pi^\alpha(x) &= \Psi(\alpha; x) \Pi^\alpha(x) = d_x \Pi_J^\alpha(x) D_J^\alpha(x) + \Pi_J^\alpha(x) d_x D_J^\alpha(x) \\ &= (d_x \Pi_J^\alpha(x) D_J^\alpha(x) + \Pi_J^\alpha(x) d_x D_J^\alpha(x)) \cdot (\Pi_J^\alpha(x) D_J^\alpha(x))^{-1} \cdot \Pi^\alpha(x) \\ &= \left[d_x \Pi_J^\alpha(x) \Pi_J^\alpha(x)^{-1} + \Pi_J^\alpha(x) \text{diag}(\alpha_J, 0, \dots, 0) \Pi_J^\alpha(x)^{-1} \frac{d_x |\tilde{x}\langle J \rangle|}{|\tilde{x}\langle J \rangle|} \right] \Pi^\alpha(x). \end{aligned}$$

Since

$$\Psi(\alpha; x) = d_x \Pi_J^\alpha(x) \Pi_J^\alpha(x)^{-1} + \Pi_J^\alpha(x) \text{diag}(\alpha_J, 0, \dots, 0) \Pi_J^\alpha(x)^{-1} \frac{d_x |\tilde{x}\langle J \rangle|}{|\tilde{x}\langle J \rangle|},$$

we have this lemma. \square

Hereafter, the matrix $M_J^{ij}(\alpha)$ is denoted by simply $M_J(\alpha)$. Let $\mathcal{M}_J(\alpha)$ be the linear transformation of $\Gamma_0(\mathcal{H}^\alpha)$ corresponding to the matrix $M_J(\alpha)$.

Lemma 3.7. *Suppose that the same assumption as Lemma 3.6.*

- (1) Let φ and φ' be an element of the eigenspace of $\mathcal{M}_J(\alpha)$ of eigenvalue α_J and that of eigenvalue 0, respectively. Then φ' represents an element of the eigenspace of $\mathcal{M}_J(-\alpha)$ of eigenvalue 0, and φ and φ' satisfy

$$\mathcal{I}(\varphi, \varphi') = 0.$$

- (2) The eigenspace of $\mathcal{M}_J(\alpha)$ of eigenvalue 0 is spanned by $\varphi\langle J' \rangle$ for $J' \in {}_{k+j}\mathcal{J}_i - \{{}_{k+j}J_i\}$.
 (3) The eigenspace of $\mathcal{M}_J(\alpha)$ of eigenvalue α_J is spanned by $\varphi\langle J \rangle$.

Proof. (1) By replacing α to $-\alpha$ for $\mathcal{M}_J(\alpha)\varphi' = 0$, we have $\mathcal{M}_J(-\alpha)\varphi' = 0$. Then φ' represents a 0-eigenvector of $\mathcal{M}_J(-\alpha)$. Proposition 3.4 together with (3.5) implies that

$$0 = d_x \mathcal{I}(\varphi, \varphi') = \mathcal{I}(\nabla_x^\alpha \varphi, \varphi') + \mathcal{I}(\varphi, \nabla_x^{-\alpha} \varphi').$$

Thus we have

$$0 = \mathcal{I}(\nabla_{ij}^\alpha \varphi, \varphi') + \mathcal{I}(\varphi, \nabla_{ij}^{-\alpha} \varphi') = \alpha_J \cdot \mathcal{I}(\varphi, \varphi') + 0 \cdot \mathcal{I}(\varphi, \varphi').$$

Since $\alpha_J \neq 0$, $\mathcal{I}(\varphi, \varphi')$ should be 0.

(2) The matrix $M_J(\alpha) = M_J^{ij}(\alpha)$ is the coefficient of $\frac{\partial}{\partial x_{ij}} \log |\tilde{x}\langle J \rangle| dx_{ij}$ by the action of $dx_{ij} \wedge \nabla_{ij}^\alpha$. Let $J' = \{j'_0 = i, j'_1, \dots, j'_k\}$ be an element of ${}_{k+j}\mathcal{J}_i$. It satisfies the assumption of Lemma 3.5. We consider the condition the factor $|\tilde{x}\langle J \rangle|$ appears in the denominator of

$$\nabla_{ij}^\alpha(\varphi\langle J' \rangle) = \alpha_{k+j} \sum_{p=0}^k \frac{|\tilde{x}\langle j'_p J'_i \rangle|}{|\tilde{x}\langle j'_p J'_{k+j} \rangle|} \varphi\langle j'_p J'_{k+j} \rangle.$$

This case only happens

$$p = 0; \quad j'_p (= j'_0) = i, \quad j'_1 = j_1, \dots, j'_k = j_k.$$

Otherwise, the factor $|\tilde{x}\langle J \rangle|$ never appears $\nabla_{ij}^\alpha(\varphi\langle J' \rangle)$ for $J' \in {}_{k+j}\mathcal{J}_i$.

(3) By Fact 3.2, we have

$$\mathcal{I}(\varphi\langle J \rangle, \varphi\langle J' \rangle) = 0$$

for any $J' \in {}_{k+j}\mathcal{J}_i - \{{}_{k+j}J_i\}$. Lemma 3.6 together with (1) implies the claim. \square

Recall that the index set

$$\begin{aligned} \dot{\mathcal{J}} = {}_{k+n+1}\mathcal{J}_0 &= \{J = \{0, j_1, \dots, j_k\} \mid 1 \leq j_1 < \dots < j_k \leq k+n\} \\ &= \{J^1 = j, J^2, \dots, J^r\} \quad \left(r = \binom{k+n}{k}\right) \end{aligned}$$

of the basis $\{\varphi\langle J \rangle\}_{J \in \dot{\mathcal{J}}}$ is aligned lexicographically, and the column vector Φ is defined as $\Phi = {}^t(\varphi\langle J^1 \rangle, \dots, \varphi\langle J^r \rangle)$. Let $C(\alpha)$ be the

intersection matrix of this basis. For any $J \in \mathcal{J}$, let v_J be the row vector defined as

$$v_J = \left(\mathcal{I}(\varphi\langle J \rangle, \varphi\langle J^1 \rangle), \dots, \mathcal{I}(\varphi\langle J \rangle, \varphi\langle J^r \rangle) \right) \cdot C(\alpha)^{-1}.$$

Then $\varphi\langle J \rangle$ is expressed as $\varphi\langle J \rangle = v_J \Phi$, and we have $v_J C(\alpha) {}^t v_{J'}^\vee = \mathcal{I}(\varphi\langle J \rangle, \varphi\langle J' \rangle)$.

Lemma 3.8. *Suppose that $\alpha_J \neq 0$. The row vector v_J is a row eigenvector of $M_J^{ij}(\alpha)$ with eigenvalue α_J . An arbitrary 0-eigenvector v of $M_J^{ij}(\alpha)$ satisfies $v C(\alpha) {}^t v_J^\vee = 0$.*

Proof. The lemma follows from Lemma 3.7 and Fact 3.2. \square

By this lemma, we obtain the following.

Proposition 3.9. *The matrix $M_J(\alpha)$ is expressed as*

$$(3.9) \quad \alpha_J C(\alpha) {}^t v_J^\vee v_J (v_J C(\alpha) {}^t v_J^\vee)^{-1} = \frac{\prod_{p \in J} \alpha_p}{(2\pi\sqrt{-1})^k} C(\alpha) {}^t v_J^\vee v_J.$$

Proof. We suppose temporarily $\alpha_J \neq 0$. Lemma 3.7 yields that the row vector v_J is a row eigenvector of $M_J^{ij}(\alpha)$ with eigenvalue α_J , and that an arbitrary 0-eigenvector v of $M_J^{ij}(\alpha)$ satisfies $v C(\alpha) {}^t v_J^\vee = 0$.

It is easy to see that the eigenspaces of $M_J(\alpha)$ coincide with those of the left hand side of (3.9). Since the factor α_J is canceled with $v_J C(\alpha) {}^t v_J^\vee$, we have the identity (3.9), which is valid even in the case $\alpha_J = 0$. \square

Remark 3.10. If we write $C(\alpha)$ as the form $(2\pi\sqrt{-1})^k C'(\alpha)$, then we can cancel the factor $(2\pi\sqrt{-1})^k$ in the right-hand side of (3.9).

Theorem 3.11. *The connection matrix $\Psi(\alpha; x)$ is expressed as*

$$(3.10) \quad \Psi(\alpha; x) = \sum_{J \in \mathcal{J}^\circ} M_J(\alpha) d_x \log |\tilde{x}\langle J \rangle|,$$

where the explicit form of $M_J(\alpha)$ is given as (3.9) in Proposition 3.9.

Proof. We have

$$\begin{aligned} \Psi(\alpha; x) &= \sum_{J \in \mathcal{J}^\circ} \left(M_J(\alpha) \sum_{i=1}^k \sum_{j \in J} \frac{\partial}{\partial x_{ij}} \log |\tilde{x}\langle J \rangle| dx_{ij} \right) \\ &= \sum_{J \in \mathcal{J}^\circ} M_J(\alpha) d_x \log |\tilde{x}\langle J \rangle|, \end{aligned}$$

by (3.7). \square

We can express the connection ∇_x^α by the intersection form. This expression is independent of choice of a frame of $\Gamma_0(\mathcal{H}^\alpha)$.

Theorem 3.12. *For any element $\varphi \in \Gamma_0(\mathcal{H}^\alpha)$, we have*

$$\begin{aligned}\nabla_x^\alpha(\varphi) &= \sum_{J \in \mathcal{J}^\circ} \alpha_J \frac{\mathcal{I}(\varphi, \varphi\langle J \rangle)}{\mathcal{I}(\varphi\langle J \rangle, \varphi\langle J \rangle)} \varphi\langle J \rangle d_x \log |\tilde{x}\langle J \rangle| \\ &= \frac{1}{(2\pi\sqrt{-1})^k} \sum_{J \in \mathcal{J}^\circ} \left(\prod_{j \in J} \alpha_j \right) \mathcal{I}(\varphi, \varphi\langle J \rangle) \varphi\langle J \rangle d_x \log |\tilde{x}\langle J \rangle|.\end{aligned}$$

Proof. Theorem 3.11 implies that ∇_x^α can be expressed as a linear combination of $d_x \log |\tilde{x}\langle J \rangle|$. We consider the linear transformation

$$\Gamma_0(\mathcal{H}^\alpha) \ni \varphi \mapsto \alpha_J \frac{\mathcal{I}(\varphi, \varphi\langle J \rangle)}{\mathcal{I}(\varphi\langle J \rangle, \varphi\langle J \rangle)} \varphi\langle J \rangle \in \Gamma_0(\mathcal{H}^\alpha).$$

By comparing the eigenspaces of this transformation with those of $\mathcal{M}_J(\alpha)$ given in Lemma 3.7, we conclude that it coincides with $\mathcal{M}_J(\alpha)$ under the condition $\alpha_J \neq 0$. Note that the factor α_J is canceled with $\mathcal{I}(\varphi\langle J \rangle, \varphi\langle J \rangle)$. Thus the connection ∇_x^α admits the expressions. \square

4. PFAFFIAN EQUATIONS

A Pfaffian equation of $F(x)$ means a first order linear differential equation for a vector-valued unknown function including $F(x)$ which is integrable and equivalent to a holonomic system of linear differential equations annihilating the single unknown function $F(x)$.

Via the equality (3.4), we can regard the equation (3.6) as a first order differential equation for a vector-valued function of rank $r = \binom{k+n}{k}$. It satisfies the integrability condition

$$d_x \Psi(\alpha; x) = \Psi(\alpha; x) \wedge \Psi(\alpha; x).$$

Thus for any point $x \in X$, there exists a unique solution to this differential equation around x under an initial condition. However, we cannot immediately regard it as a Pfaffian equation of

$$F(\alpha; x) = \int_{\square} \prod_{j=1}^{n+k+1} L_j^{\alpha_j} \varphi\langle \dot{J} \rangle,$$

where $\dot{J} = \{0, 1, \dots, k\}$ and \square is a twisted cycle. To obtain differential equations annihilating $F(\alpha; x)$ from it, we need to express

$$\int_{\square} \prod_{j=1}^{n+k+1} L_j^{\alpha_j} \varphi\langle J \rangle \quad (J \in {}_{k+n+1}\mathcal{J}_0)$$

by actions of the ring of differential operators with rational function coefficients $\mathbb{C}(\dots, x_{ij}, \dots) \langle \dots, \frac{\partial}{\partial x_{ij}}, \dots \rangle$ on $F(\alpha; x)$. In this section, by differentiating $F(\alpha; x)$ several times, we find a vector-valued function $\mathbf{F}(\alpha; x)$ such that it satisfies a Pfaffian equation with the connection matrix $\Psi(\alpha; x)$.

By the equality of (3.3), we can translate computations of $\frac{\partial}{\partial x_{ij}} F(\alpha; x)$ to those of $\nabla_{ij}^\alpha(\varphi\langle J \rangle)$. Firstly, we express $\nabla_{ij}^\alpha(\varphi\langle J \rangle)$ in terms of $\varphi\langle J \rangle$. Since $(\partial/\partial x_{ij})(\varphi\langle J \rangle) = 0$, we have

$$\nabla_{ij}^\alpha(\varphi\langle J \rangle) = \frac{\alpha_{k+j} dt}{L_{k+j} t_1 \cdots t_{i-1} t_{i+1} \cdots t_k} = \frac{\alpha_{k+j}}{|\tilde{x}\langle {}_i \dot{J}_{k+j} \rangle|} \varphi\langle {}_i \dot{J}_{k+j} \rangle,$$

where

$${}_i \dot{J}_{k+j} = (J - \{i\}) \cup \{k+j\} = \{0, 1, \dots, i-1, k+j, i+1, \dots, k\}.$$

By these operators, we obtain $k \times n$ functions.

Secondly, we express $(\nabla_{i'j'}^\alpha \circ \nabla_{ij}^\alpha)(\varphi\langle J \rangle)$ in terms of $\varphi\langle J \rangle$, where $i \neq i'$ and $j \neq j'$. We have

$$\begin{aligned} & (\nabla_{i'j'}^\alpha \circ \nabla_{ij}^\alpha)(\varphi\langle J \rangle) \\ &= \nabla_{i'j'}^\alpha \left(\frac{\alpha_{k+j} dt}{L_{k+j} t_1 \cdots t_{i-1} t_{i+1} \cdots t_k} \right) \\ &= \frac{\alpha_{k+j'} \alpha_{k+j} dt}{L_{k+j'} L_{k+j} t_1 \cdots t_{i-1} t_{i+1} \cdots t_{i'-1} t_{i'+1} \cdots t_k} \\ &= \frac{\alpha_{k+j} \alpha_{k+j'}}{|\tilde{x}\langle {}_{i',i} \dot{J}_{k+j,k+j'} \rangle|} \varphi\langle {}_{i',i} \dot{J}_{k+j,k+j'} \rangle, \end{aligned}$$

where ${}_{i',i} \dot{J}_{k+j,k+j'} = (J - \{i, i'\}) \cup \{k+j, k+j'\}$. Since

$$\nabla_{i'j}^\alpha \circ \nabla_{ij'}^\alpha = \nabla_{ij'}^\alpha \circ \nabla_{i'j}^\alpha = \nabla_{ij}^\alpha \circ \nabla_{i'j'}^\alpha = \nabla_{i'j'}^\alpha \circ \nabla_{ij}^\alpha,$$

we obtain $\binom{k}{2} \times \binom{n}{2}$ functions, which is the number of the way to choose i_1, i_2, j_1, j_2 such that $1 \leq i_1 < i_2 \leq k$, $1 \leq j_1 < j_2 \leq n$.

Thirdly, we act $\nabla_{i_3 j_3}^\alpha \circ \nabla_{i_2 j_2}^\alpha \circ \nabla_{i_1 j_1}^\alpha$ on $\varphi\langle J \rangle$, where

$$\{i_1, i_2, i_3\} \subset \{1, \dots, k\}, \quad \{j_1, j_2, j_3\} \subset \{1, \dots, n\},$$

and their cardinalities are 3. We have

$$\begin{aligned}
& (\nabla_{i_3 j_3}^\alpha \circ \nabla_{i_2 j_2}^\alpha \circ \nabla_{i_1 j_1}^\alpha)(\varphi\langle \dot{J} \rangle) \\
&= \frac{\alpha_{k+j_1} \alpha_{k+j_2} \alpha_{k+j_3} t_{i_1} t_{i_2} t_{i_3} dt}{L_{k+j_1} L_{k+j_2} L_{k+j_3} t_1 \cdots t_k} \\
&= \frac{\alpha_{k+j_1} \alpha_{k+j_2} \alpha_{k+j_3}}{|\tilde{x}\langle_{i_3, i_2, i_1} \dot{J}_{k+j_1, k+j_2, k+j_3} \rangle|} \varphi\langle_{i_3, i_2, i_1} \dot{J}_{k+j_1, k+j_2, k+j_3} \rangle,
\end{aligned}$$

where $_{i_3, i_2, i_1} \dot{J}_{k+j_1, k+j_2, k+j_3} = (\dot{J} - \{i_1, i_2, i_3\}) \cup \{k+j_1, k+j_2, k+j_3\}$. By these operators, we obtain $\binom{k}{3} \times \binom{n}{3}$ functions, which is the number of the way to choose $i_1, i_2, i_3, j_1, j_2, j_3$ such that $1 \leq i_1 < i_2 < i_3 \leq k$, $1 \leq j_1 < j_2 < j_3 \leq n$.

Generally, we have

$$(4.1) \quad (\nabla_{i_l j_l}^\alpha \circ \cdots \circ \nabla_{i_1 j_1}^\alpha)(\varphi\langle \dot{J} \rangle) = \frac{\alpha_{k+j_1} \cdots \alpha_{k+j_l}}{|\tilde{x}\langle_{i_l, \dots, i_1} \dot{J}_{k+j_1, \dots, k+j_l} \rangle|} \varphi\langle_{i_l, \dots, i_1} \dot{J}_{k+j_1, \dots, k+j_l} \rangle,$$

where $_{i_l, \dots, i_1} \dot{J}_{k+j_1, \dots, k+j_l} = (\dot{J} - \{i_1, \dots, i_l\}) \cup \{k+j_1, \dots, k+j_l\}$. Note that

$$0 \in _{i_l, \dots, i_1} \dot{J}_{k+j_1, \dots, k+j_l}, \quad k+n+1 \notin _{i_l, \dots, i_1} \dot{J}_{k+j_1, \dots, k+j_l}.$$

In this way, we have

$$\sum_{i=0}^k \binom{k}{i} \times \binom{n}{i} = \binom{k+n}{k}$$

functions. The set of $_{i_l, \dots, i_1} \dot{J}_{k+j_1, \dots, k+j_l}$'s coincides with the set

$$\dot{\mathcal{J}} = {}_{k+n+1} \mathcal{J}_0 = \{J = \{0, j_1, \dots, j_k\} \mid 1 \leq j_1 < \cdots < j_k \leq k+n\}.$$

Recall that they are aligned lexicographically $\dot{\mathcal{J}} = \{J^1 = \dot{J}, J^2, \dots, J^r\}$, $r = \binom{k+n}{k}$. Recall also that J^p is expressed as $\{0, j_1, \dots, j_k\}$ with $1 \leq j_1 < \cdots < j_k \leq n+k$. Note that if $J^p = _{i_l, \dots, i_1} \dot{J}_{k+j_1, \dots, k+j_l}$ as sets, we have

$$\varphi\langle J^p \rangle = \text{sgn} \left(\begin{matrix} J^p \\ _{i_l, \dots, i_1} \dot{J}_{k+j_1, \dots, k+j_l} \end{matrix} \right) \cdot \varphi\langle_{i_l, \dots, i_1} \dot{J}_{k+j_1, \dots, k+j_l} \rangle.$$

For example, if $k=2$, we have $\varphi\langle 023 \rangle = -\varphi\langle 032 \rangle = -\varphi\langle_1 \dot{J}_3 \rangle$.

We define a vector-valued function $\mathbf{F}(\alpha; x)$ by

$$\mathbf{F}(\alpha; x) = \begin{pmatrix} \int_{\square} \prod_{j=1}^{k+n+1} L_j^{\alpha_j} \varphi\langle \dot{J} \rangle \\ \int_{\square} \prod_{j=1}^{k+n+1} L_j^{\alpha_j} \varphi\langle J^2 \rangle \\ \vdots \\ \int_{\square} \prod_{j=1}^{k+n+1} L_j^{\alpha_j} \varphi\langle J^r \rangle \end{pmatrix} = \int_{\square} \prod_{j=1}^{k+n+1} L_j^{\alpha_j} \cdot \Phi.$$

By (3.3) and (4.1), it is expressed as

$$\mathbf{F}(\alpha; x) = G(\alpha; x) \tilde{\mathbf{F}}(\alpha; x),$$

where

$$\begin{aligned} \tilde{\mathbf{F}}(\alpha; x) &= {}^t \left(F(\alpha; x), \frac{\partial F(\alpha; x)}{\partial x_{k1}}, \dots, \frac{\partial^l F(\alpha; x)}{\partial x_{i_1 j_1} \cdots \partial x_{i_l j_l}}, \dots \right), \\ G(\alpha; x) &= \text{diag} \left(1, \frac{x_{k1}}{\alpha_{k+1}}, \dots, \pm \frac{|\tilde{x}_{\langle i_l, \dots, i_1 \rangle J_{k+j_1, \dots, k+j_l}}|}{\prod_{s=1}^l \alpha_{k+j_s}}, \dots \right). \end{aligned}$$

Note that $G(\alpha; x)$ belongs to $GL(r; \Omega^0(X))$. We call the vector-valued function $\mathbf{F}(\alpha; x)$ (resp. $\tilde{\mathbf{F}}(\alpha; x)$) the *Gauss-Manin vector* of $F(\alpha; x)$ with respect to the frame $\{\varphi\langle J \rangle\}_{J \in \mathcal{J}}$ (resp. $G(\alpha; x)^{-1} \{\varphi\langle J \rangle\}_{J \in \mathcal{J}}$), or shortly the *G-M vector* of $F(\alpha; x)$. Here, $G(\alpha; x)^{-1} \{\varphi\langle J \rangle\}_{J \in \mathcal{J}}$ means the frame corresponding to the vector $G(\alpha; x)^{-1} \Phi$.

Theorem 4.1. *The G-M vector $\mathbf{F}(\alpha; x)$ of $F(\alpha; x)$ satisfies the Pfaffian equation*

$$d_x \mathbf{F}(\alpha; x) = \Psi(\alpha; x) \mathbf{F}(\alpha; x)$$

with the same connection matrix as in (3.10).

Proof. Since $\mathbf{F}(\alpha; x)$ is defined as $\int_{\square} \prod_{j=1}^{k+n+1} L_j^{\alpha_j} \cdot \Phi$, it satisfies $d_x \mathbf{F}(\alpha; x) = \Psi(\alpha; x) \mathbf{F}(\alpha; x)$. It is clear that this equation is equivalent to a rank $\binom{k+n}{k}$ system of differential equations annihilating $F(\alpha; x)$. \square

Corollary 4.2. *The G-M vector $\tilde{\mathbf{F}}(\alpha; x)$ satisfies the Pfaffian equation*

$$d_x \tilde{\mathbf{F}}(\alpha; x) = \tilde{\Psi}(\alpha; x) \tilde{\mathbf{F}}(\alpha; x)$$

with the connection matrix

$$\tilde{\Psi}(\alpha; x) = G(\alpha; x)^{-1} \Psi(\alpha; x) G(\alpha; x) + d_x G(\alpha; x)^{-1} G(\alpha; x).$$

Proof. We see the expression of $\tilde{\Psi}(\alpha; x)$. By Theorem 4.1, the G-M vector $\tilde{\mathbf{F}}(\alpha; x) = G(\alpha; x)^{-1} \mathbf{F}(\alpha; x)$ satisfies

$$\begin{aligned} d_x \tilde{\mathbf{F}} &= d_x (G^{-1} \mathbf{F}) = (d_x G^{-1}) \mathbf{F} + G^{-1} d_x \mathbf{F} \\ &= (d_x G^{-1})(G G^{-1}) \mathbf{F} + G^{-1} \Psi \mathbf{F} = [d_x G^{-1} G + G^{-1} \Psi G] \tilde{\mathbf{F}}, \end{aligned}$$

where $\mathbf{F} = \mathbf{F}(\alpha; x)$, $\Psi = \Psi(\alpha, x)$ and $G = G(\alpha, x)$. \square

5. CONTIGUITY RELATIONS

In this section, we give contiguity relations of $F(\alpha; x)$ by using linear maps on twisted cohomology groups and intersection forms.

For $i = 1, \dots, k + n + 1$, we consider a linear map

$$\Omega^l(T_x) \ni \varphi \mapsto L_i \cdot \varphi \in \Omega^l(T_x).$$

We put $\alpha^{(i)} := (\alpha_0 - 1, \dots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \dots, \alpha_{k+n+1})$, $\omega^{(i)} := \omega + d_t \log L_i$, and $\nabla^{\alpha^{(i)}} = d_t + \omega^{(i)} \wedge$.

Notation 5.1. In this section, we write

$$\begin{aligned} V &= H^k(\Omega^\bullet(T_x), \nabla^\alpha), & V^{(i)} &= H^k(\Omega^\bullet(T_x), \nabla^{\alpha^{(i)}}), \\ V^\vee &= H^k(\Omega^\bullet(T_x), \nabla^{-\alpha}), & V^{(i)\vee} &= H^k(\Omega^\bullet(T_x), \nabla^{-\alpha^{(i)}}), \end{aligned}$$

for simplicity. For a given $\psi \in \Omega^k(T_x)$, to clarify which cohomology group ψ belongs to, we denote by $[\psi]$, $[\psi]_i$, $[\psi]^\vee$, and $[\psi]_i^\vee$ the element of V , $V^{(i)}$, V^\vee , and $V^{(i)\vee}$ represented by ψ , respectively.

Proposition 5.2. *The map*

$$\mathcal{U}_i : V^{(i)} \ni [\varphi]_i \mapsto [L_i \cdot \varphi] \in V$$

is a well-defined linear map.

Proof. Since we have

$$\begin{aligned} L_i \nabla^{\alpha^{(i)}}(\varphi) &= L_i \cdot (d_t \varphi + \omega^{(i)} \wedge \varphi) = L_i \cdot \left(d_t \varphi + \omega \wedge \varphi + \frac{d_t L_i}{L_i} \wedge \varphi \right) \\ &= d_t(L_i \cdot \varphi) + \omega \wedge (L_i \cdot \varphi) = \nabla^\alpha(L_i \varphi), \end{aligned}$$

the L_i -multiplication descends to a map from $V^{(i)}$ to V . \square

Let $U_i(\alpha; x)$ be the representation matrix of \mathcal{U}_i with respect to the bases $\{[\varphi\langle J \rangle]_i\}_{J \in \mathcal{J}}$ of $V^{(i)}$ and $\{[\varphi\langle J \rangle]\}_{J \in \mathcal{J}}$ of V . Recall that $\mathbf{F}(\alpha; x)$ is a vector valued function defined as

$${}^t \left(\int_{\square} \prod_{j=1}^{n+k+1} L_j^{\alpha_j} \varphi\langle J \rangle, \int_{\square} \prod_{j=1}^{n+k+1} L_j^{\alpha_j} \varphi\langle J^2 \rangle, \dots, \int_{\square} \prod_{j=1}^{n+k+1} L_j^{\alpha_j} \varphi\langle J^r \rangle \right).$$

Since

$$\int_{\square} \prod_{j=1}^{k+n+1} L_j^{\alpha_j} \cdot \mathcal{U}_i([\varphi\langle J \rangle]_i) = \int_{\square} \prod_{j=1}^{k+n+1} L_j^{\alpha_j} \cdot L_i \cdot \varphi\langle J \rangle = F(\alpha^{(i)}; x),$$

we have the contiguity relation

$$\mathbf{F}(\alpha^{(i)}; x) = U_i(\alpha; x) \cdot \mathbf{F}(\alpha; x).$$

We give an explicit expression of $U_i(\alpha; x)$.

Theorem 5.3. *The representation matrix $U_i(\alpha; x)$ admits the expression*

$$U_i(\alpha; x) = C(\alpha^{(i)})P_i(\alpha^{(i)})^{-1}D_i(x)Q_i(\alpha)C(\alpha)^{-1},$$

where

$$D_i(x) = \text{diag} \left(\dots, \frac{|\tilde{x}\langle J \rangle|}{|\tilde{x}\langle {}_i J_0 \rangle|}, \dots \right)_{J \in {}_0 \mathcal{J}_i}, \quad C(\alpha) = \left(\mathcal{I}(\varphi\langle I \rangle, \varphi\langle J \rangle) \right)_{I, J \in \dot{\mathcal{J}}},$$

$$P_i(\alpha) = \left(\mathcal{I}(\varphi\langle I \rangle, \varphi\langle J \rangle) \right)_{I \in {}_0 \mathcal{J}_i, J \in \dot{\mathcal{J}}}, \quad Q_i(\alpha) = \left(\mathcal{I}(\varphi\langle I \rangle, \varphi\langle J \rangle) \right)_{I \in {}_i \mathcal{J}_0, J \in \dot{\mathcal{J}}}.$$

Remark 5.4. (1) We explain how to align the elements of ${}_{k+n+1}\mathcal{J}_0 (= \dot{\mathcal{J}})$, ${}_0\mathcal{J}_i$, and ${}_i\mathcal{J}_0$. First, recall that we align the elements of ${}_{k+n+1}\mathcal{J}_0$ lexicographically, and denote them ${}_{k+n+1}\mathcal{J}_0 = \{J^1, \dots, J^r\}$, where $r = \binom{k+n}{k}$. Next, we align the elements of ${}_i\mathcal{J}_0$ as

$${}_i\mathcal{J}_0 = \{J'^1, \dots, J'^r\}, \quad J'^l = \begin{cases} J^l & (i \notin J^l), \\ {}_i J_{k+n+1}^l & (i \in J^l). \end{cases}$$

Finally, we align the elements of ${}_0\mathcal{J}_i$ as

$${}_0\mathcal{J}_i = \{{}_0 J_i'^1, \dots, {}_0 J_i'^r\}.$$

For example, if $k = n = 2$ and $i = 3$, then

$$\begin{aligned} {}_5\mathcal{J}_0 &= \{\{012\}, \{013\}, \{014\}, \{023\}, \{024\}, \{034\}\}, \\ {}_3\mathcal{J}_0 &= \{\{012\}, \{015\}, \{014\}, \{025\}, \{025\}, \{054\}\}, \\ {}_0\mathcal{J}_3 &= \{\{312\}, \{315\}, \{314\}, \{325\}, \{325\}, \{354\}\}. \end{aligned}$$

(2) The intersection numbers $\mathcal{I}(\varphi\langle I \rangle, \varphi\langle J \rangle)$ in the theorem can be computed by Fact 3.2. We denote the intersection pairing between $V^{(i)}$ and $V^{(i)\vee}$ by $\mathcal{I}^{(i)}$. Then, it is easy to see that

$$\begin{aligned} \left(\mathcal{I}^{(i)}([\varphi\langle I \rangle]_i, [\varphi\langle J \rangle]_i^\vee) \right)_{I, J \in \dot{\mathcal{J}}} &= C(\alpha^{(i)}), \\ \left(\mathcal{I}^{(i)}([\varphi\langle I \rangle]_i, [\varphi\langle J \rangle]_i^\vee) \right)_{I \in {}_0 \mathcal{J}_i, J \in \dot{\mathcal{J}}} &= P_i(\alpha^{(i)}). \end{aligned}$$

Proof of Theorem 5.3. Recall that $L_0 = 1$. For $J \in {}_0\mathcal{J}_i$, because of

$$\varphi\langle J \rangle = |\tilde{x}\langle J \rangle| \cdot \frac{dt}{\prod_{j \in J} L_j} = |\tilde{x}\langle J \rangle| \cdot \frac{dt}{L_i \cdot \prod_{j \in {}_i J_0} L_j},$$

we have

$$L_i \cdot \varphi\langle J \rangle = |\tilde{x}\langle J \rangle| \cdot \frac{dt}{\prod_{j \in {}_i J_0} L_j} = \frac{|\tilde{x}\langle J \rangle|}{|\tilde{x}\langle {}_i J_0 \rangle|} \cdot \varphi\langle {}_i J_0 \rangle.$$

Hence, the alignment mentioned in Remark 5.4 (1) means that the representation matrix of \mathcal{U}_i with respect to the bases

$$\{[\varphi\langle J\rangle]_i\}_{J\in {}_0\mathcal{J}_i}\subset V^{(i)}, \quad \{[\varphi\langle J\rangle]\}_{J\in {}_i\mathcal{J}_0}\subset V$$

coincides with $D_i(x)$. By linearity of the intersection forms \mathcal{I} and $\mathcal{I}^{(i)}$, we can show that

$$\begin{pmatrix} [\varphi\langle J'^1\rangle] \\ \vdots \\ [\varphi\langle J''^r\rangle] \end{pmatrix} = Q_i(\alpha)C(\alpha)^{-1} \begin{pmatrix} [\varphi\langle J^1\rangle] \\ \vdots \\ [\varphi\langle J^r\rangle] \end{pmatrix},$$

$$\begin{pmatrix} [\varphi\langle {}_0J_i^1\rangle]_i \\ \vdots \\ [\varphi\langle {}_0J_i^r\rangle]_i \end{pmatrix} = P_i(\alpha^{(i)})C(\alpha^{(i)})^{-1} \begin{pmatrix} [\varphi\langle J^1\rangle]_i \\ \vdots \\ [\varphi\langle J^r\rangle]_i \end{pmatrix}.$$

These imply that the representation matrix $U_i(\alpha; x)$ coincides with $C(\alpha^{(i)})P_i(\alpha^{(i)})^{-1}D_i(x)Q_i(\alpha)C(\alpha)^{-1}$. \square

In the remainder of this section, we consider relations between the linear map \mathcal{U}_i and the intersection form \mathcal{I} .

Theorem 5.5. *The linear map $\mathcal{U}_i : V^{(i)} \rightarrow V$ is expressed as*

$$\begin{aligned} (5.1) \quad \mathcal{U}_i([\varphi]_i) &= \sum_{J\in {}_0\mathcal{J}_i} \frac{\mathcal{I}^{(i)}([\varphi]_i, [\varphi\langle {}_iJ_0\rangle]_i^\vee)}{\mathcal{I}^{(i)}([\varphi\langle J\rangle]_i, [\varphi\langle {}_iJ_0\rangle]_i^\vee)} \cdot \frac{|\tilde{x}\langle J\rangle|}{|\tilde{x}\langle {}_iJ_0\rangle|} \cdot [\varphi\langle {}_iJ_0\rangle] \\ &= \sum_{J\in {}_i\mathcal{J}_0} \frac{\mathcal{I}^{(i)}([\varphi]_i, [\varphi\langle J\rangle]_i^\vee)}{\mathcal{I}^{(i)}([\varphi\langle {}_0J_i\rangle]_i, [\varphi\langle J\rangle]_i^\vee)} \cdot \frac{|\tilde{x}\langle {}_0J_i\rangle|}{|\tilde{x}\langle J\rangle|} \cdot [\varphi\langle J\rangle]. \end{aligned}$$

Proof. Since a correspondence ${}_0\mathcal{J}_i \ni J \mapsto {}_iJ_0 \in {}_i\mathcal{J}_0$ is one-to-one, the second equality is clear. We show the first one. By Fact 3.2, we have

$$\mathcal{I}^{(i)}([\varphi\langle J\rangle]_i, [\varphi\langle {}_iJ'_0\rangle]_i^\vee) = \begin{cases} \frac{(2\pi\sqrt{-1})^k}{\prod_{j\in J-\{i\}} \alpha_j} & \text{if } J = J', \\ 0 & \text{otherwise,} \end{cases}$$

for $J, J' \in {}_0\mathcal{J}_i$. If $\varphi = \varphi\langle J\rangle$ with $J \in {}_0\mathcal{J}_i$, then the right-hand side of (5.1) is

$$\frac{|\tilde{x}\langle J\rangle|}{|\tilde{x}\langle {}_iJ_0\rangle|} \cdot [\varphi\langle {}_iJ_0\rangle]$$

which is nothing but $\mathcal{U}_i([\varphi\langle J\rangle]_i)$, by the proof of Theorem 5.3. Since $\{[\varphi\langle J\rangle]_i\}_{J\in {}_0\mathcal{J}_i}$ form a basis of $V^{(i)}$, the first equality holds. \square

In a way similar to that used in Proposition 5.2 and Theorem 5.5, we can show the following.

Corollary 5.6. *The inverse map of \mathcal{U}_i is given by a well-defined map*

$$\mathcal{U}_i^{-1} : V \ni [\varphi] \mapsto \left[\frac{1}{L_i} \cdot \varphi \right]_i \in V^{(i)}.$$

It also admits the expression

$$(5.2) \quad \mathcal{U}_i^{-1}([\varphi]) = \sum_{J \in \mathcal{J}_i} \frac{\mathcal{I}([\varphi], [\varphi \langle J \rangle]^\vee)}{\mathcal{I}([\varphi \langle J_0 \rangle], [\varphi \langle J \rangle]^\vee)} \cdot \frac{|\tilde{x} \langle J_0 \rangle|}{|\tilde{x} \langle J \rangle|} \cdot [\varphi \langle J \rangle]_i.$$

Remark 5.7. Expressions similar to (5.1) and (5.2) are given in [AK, §4.4.2] without the intersection forms. Their calculations are done in only the target space V (resp. $V^{(i)}$) of \mathcal{U}_i (resp. \mathcal{U}_i^{-1}), and are complicated. On the other hand, by considering not only the target spaces but also the domains of \mathcal{U}_i and \mathcal{U}_i^{-1} , we can obtain a simple structure of contiguity relations.

Replacing α by $-\alpha^{(i)}$ in Proposition 5.2, we obtain the linear map

$$\mathcal{U}_i^\vee : V^\vee \ni [\varphi]^\vee \mapsto [L_i \cdot \varphi]_i^\vee \in V^{(i)\vee}.$$

Proposition 5.8. *For any $[\varphi]_i \in V^{(i)}$ and $[\psi]^\vee \in V^\vee$, we have*

$$\mathcal{I}(\mathcal{U}_i([\varphi]_i), [\psi]^\vee) = \mathcal{I}^{(i)}([\varphi]_i, \mathcal{U}_i^\vee([\psi]^\vee)).$$

Proof. By [M1], there exist C^∞ k -forms φ' and η on T_x such that the support of φ' is compact and

$$\varphi = \varphi' + \nabla^{\alpha^{(i)}} \eta.$$

By the proof of Proposition 5.2, we have

$$L_i \cdot \varphi = L_i \cdot \varphi' + \nabla^\alpha (L_i \cdot \eta).$$

Since the support of $L_i \cdot \varphi'$ is also compact, the intersection numbers are expressed as

$$\begin{aligned} \mathcal{I}(\mathcal{U}_i([\varphi]_i), [\psi]^\vee) &= \int_{T_x} (L_i \cdot \varphi') \wedge \psi, \\ \mathcal{I}^{(i)}([\varphi]_i, \mathcal{U}_i^\vee([\psi]^\vee)) &= \int_{T_x} \varphi' \wedge (L_i \cdot \psi). \end{aligned}$$

Obviously, these two integrations coincide. \square

By considering the bases of V , V^\vee , $V^{(i)}$, and $V^{(i)\vee}$ represented by $\{\varphi \langle J \rangle\}_{J \in \mathcal{J}}$, we obtain the following identity.

Corollary 5.9. $U_i(\alpha; x)C(\alpha) = C(\alpha^{(i)}) {}^t U_i(-\alpha^{(i)}; x).$

6. RELATIONS FOR HYPERGEOMETRIC SERIES

For applications to algebraic statistics, we need to reduce our formulas for hypergeometric integrals to those for hypergeometric series. In this section, we specialize a twisted cycle \square to $\Delta\langle 1, \dots, k, k+n+1 \rangle$, which is the regularization of the standard simplex (3.8) with respect to $\prod_{j=1}^{k+n+1} L_j^{\alpha_j}$. Then the integral $F(\alpha; x)$ admits a power series expansion for x sufficiently close to the zero matrix O . We give relations between this expansion and the hypergeometric series defined in [AK].

We put

$$S(\alpha; x) = \sum_{m=(m_{ij}) \in M(k, n; \mathbb{Z}_{\geq 0})} \frac{1}{\Gamma_m(\alpha)} \cdot \prod_{i,j} x_{ij}^{m_{ij}},$$

where

$$\begin{aligned} \Gamma_m(\alpha) = & \prod_{i=1}^k \Gamma(-\alpha_i - \sum_{j=1}^n m_{ij} + 1) \cdot \prod_{j=1}^n \Gamma(\alpha_{k+j} - \sum_{i=1}^k m_{ij} + 1) \\ & \cdot \Gamma(\sum_{i=1}^k \alpha_i + \alpha_{k+n+1} + \sum_{i=1}^k \sum_{j=1}^n m_{ij} + 1) \cdot \prod_{i=1}^k \prod_{j=1}^n \Gamma(m_{ij} + 1). \end{aligned}$$

Note that $S(\alpha; x)$ coincides with the hypergeometric series of type $(k+1, k+n+2)$ defined in [AK, §3.1.3], modulo gamma factors.

Proposition 6.1. *We specialize a twisted cycle \square to $\Delta\langle 1, \dots, k, k+n+1 \rangle$. If each x_{ij} is sufficiently close to 0, then the integral $F(\alpha; x)$ admits the power series expansion*

$$e^{-\pi\sqrt{-1}(\alpha_1+\dots+\alpha_k)} \cdot \prod_{i=1}^k \Gamma(\alpha_i) \Gamma(-\alpha_i + 1) \cdot \prod_{j=1}^{n+1} \Gamma(\alpha_{k+j} + 1) \cdot S(\alpha; x).$$

Proof. We give the arguments of L_i on the standard simplex Δ in (3.8) as follows.

	$i = 1, \dots, k$	$i = k+1, \dots, k+n$	$i = k+n+1$
$\arg L_i$	$-\pi$	0	0

By putting $t_i = e^{-\pi\sqrt{-1}} s_i (= -s_i)$ in the integration

$$F(\alpha; x) = \int_{\Delta\langle 1, \dots, k, k+n+1 \rangle} \prod_{j=1}^{k+n+1} L_j^{\alpha_j} \varphi\langle j \rangle,$$

we can show the proposition in an analogous way used in [AK, §3.3]. \square

We put

$$\begin{aligned} \mathbf{S}(\alpha; x) &= e^{\pi\sqrt{-1}(\alpha_1+\dots+\alpha_k)} \cdot \prod_{i=1}^k \frac{1}{\Gamma(\alpha_i)\Gamma(-\alpha_i+1)} \cdot \prod_{j=1}^{n+1} \frac{1}{\Gamma(\alpha_{k+j}+1)} \cdot \mathbf{F}(\alpha, x) \\ &= \begin{pmatrix} S(\alpha; x) \\ \vdots \\ \pm \frac{|\tilde{x}\langle i_l, \dots, i_1, j_{k+j_1}, \dots, k+j_{j_l} \rangle|}{\prod_{s=1}^l \alpha_{k+j_s}} \cdot \frac{\partial^l S(\alpha; x)}{\partial x_{i_1 j_1} \cdots \partial x_{i_l j_l}} \\ \vdots \end{pmatrix}, \end{aligned}$$

which is the G-M vector of $S(\alpha; x)$. We consider the Pfaffian equation and the contiguity relations with respect to $\mathbf{S}(\alpha; x)$.

Corollary 6.2. $d_x \mathbf{S}(\alpha; x) = \Psi(\alpha; x) \mathbf{S}(\alpha; x)$.

Proof. $\mathbf{S}(\alpha; x)$ is defined as a scalar multiple of $\mathbf{F}(\alpha; x)$ and this scalar is independent of x . \square

Corollary 6.3. For $1 \leq i \leq k$ and $1 \leq j \leq n+1$, we have

$$\begin{aligned} \mathbf{S}(\alpha^{(i)}; x) &= U_i(\alpha; x) \mathbf{S}(\alpha; x), \\ \mathbf{S}(\alpha^{(k+j)}; x) &= \frac{1}{\alpha_{k+j}+1} U_{k+j}(\alpha; x) \mathbf{S}(\alpha; x). \end{aligned}$$

Proof. By

$$(-1) \cdot \frac{\Gamma(\alpha_i)\Gamma(-\alpha_i+1)}{\Gamma(\alpha_i+1)\Gamma(-\alpha_i)} = 1, \quad \frac{\Gamma(\alpha_{k+j}+1)}{\Gamma(\alpha_{k+j}+2)} = \frac{1}{\alpha_{k+j}+1},$$

and Theorem 5.3, we have the identities. \square

7. NORMALIZING CONSTANTS OF TWO-WAY CONTINGENCY TABLES

We apply contiguity relations and the Pfaffian equation to the numerical evaluation of the normalizing constants of the hypergeometric distribution of the $r_1 \times r_2$ contingency tables with fixed marginal sums. In this section, we explain how our results are applied, and give an algorithm that evaluates the normalizing constants.

We consider an $r_1 \times r_2$ contingency table

$$u = \begin{array}{c} \begin{array}{|c|c|c|c|} \hline u_{11} & u_{12} & \cdots & u_{1r_2} \\ \hline u_{21} & u_{22} & \cdots & u_{2r_2} \\ \hline \vdots & \vdots & & \vdots \\ \hline u_{r_1 1} & u_{r_1 2} & \cdots & u_{r_1 r_2} \\ \hline \end{array} \begin{array}{l} \beta_1^{(1)} \\ \beta_2^{(1)} \\ \vdots \\ \beta_{r_1}^{(1)} \end{array} \\ \begin{array}{ccc} \beta_1^{(2)} & \beta_2^{(2)} & \cdots & \beta_{r_2}^{(2)} \end{array} \end{array} \quad u_{ij} \in \mathbb{Z}_{\geq 0}.$$

Here, $\beta_i^{(1)} := \sum_{j=1}^{r_2} u_{ij}$ is the row sum, and $\beta_j^{(2)} := \sum_{i=1}^{r_1} u_{ij}$ is the column sum. For fixed marginal sums $\beta = (\beta^{(1)}; \beta^{(2)}) = (\beta_1^{(1)}, \dots, \beta_{r_1}^{(1)}; \beta_1^{(2)}, \dots, \beta_{r_2}^{(2)})$ and a variable matrix $p = (p_{ij})_{1 \leq i \leq r_1, 1 \leq j \leq r_2}$, the polynomial

$$Z(\beta; p) = \sum_u \frac{p^u}{u!} = \sum_u \frac{\prod_{i,j} p_{ij}^{u_{ij}}}{\prod_{i,j} u_{ij}!} = \sum_u \frac{\prod_{i,j} p_{ij}^{u_{ij}}}{\prod_{i,j} \Gamma(u_{ij} + 1)}$$

in p_{ij} is called the normalizing constant, where the sum is taken over all contingency tables u with marginal sums β .

Proposition 7.1. *We put the parameters and variables in $S(\alpha; x)$ as follows:*

$$\begin{aligned} (k, n) &:= (r_1 - 1, r_2 - 1), \\ \alpha &= (\alpha_0, \dots, \alpha_{k+n+1}) = (\alpha_0, \alpha_1, \dots, \alpha_{r_1-1}, \alpha_{r_1}, \dots, \alpha_{r_1+r_2-2}, \alpha_{r_1+r_2-1}) \\ &:= (-\beta_{r_1}^{(1)}, -\beta_1^{(1)}, \dots, -\beta_{r_1-1}^{(1)}, \beta_2^{(2)}, \dots, \beta_{r_2}^{(2)}, \beta_1^{(2)}), \\ x &:= (x_{ij})_{1 \leq i \leq k, 1 \leq j \leq n}, \quad x_{ij} = \frac{p_{i,j+1} p_{r_1,1}}{p_{i1} p_{r_1,j+1}}. \end{aligned}$$

Then the normalizing constant is expressed as

$$Z(\beta; p) = \prod_{i=1}^k p_{i1}^{-\alpha_i} \cdot \prod_{j=1}^n p_{r_1,j+1}^{\alpha_{k+j}} \cdot p_{r_1,1}^{\sum_{i=1}^k \alpha_i + \alpha_{k+n+1}} \cdot S(\alpha; x).$$

Note that $S(\alpha; x)$ in the right-hand side is a polynomial in x .

Proof. We put

$$u_0 := \begin{array}{|c|c|c|c|} \hline \beta_1^{(1)} & 0 & \cdots & 0 \\ \hline \vdots & \vdots & & \vdots \\ \hline \beta_{r_1-1}^{(1)} & 0 & \cdots & 0 \\ \hline \beta_1^{(2)} - \sum_{i=1}^{r_1-1} \beta_i^{(1)} & \beta_2^{(2)} & \cdots & \beta_{r_2}^{(2)} \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline -\alpha_1 & 0 & \cdots & 0 \\ \hline \vdots & \vdots & & \vdots \\ \hline -\alpha_k & 0 & \cdots & 0 \\ \hline \alpha_{k+n+1} + \sum_{i=1}^k \alpha_i & \alpha_{k+1} & \cdots & \alpha_{k+n} \\ \hline \end{array},$$

$$\ell_{ij} := \begin{array}{|c|c|c|c|} \hline \vdots & & \vdots & \\ \hline -1 & \cdots & 1 & \cdots \\ \hline \vdots & & \vdots & \\ \hline 1 & \cdots & -1 & \cdots \\ \hline \end{array} \quad \begin{array}{c} i \\ j+1 \end{array} \quad (\text{the other entries are } 0).$$

The contingency table u with the marginal sums β is expressed as

$$u = u_0 + \sum_{i=1}^{r_1-1} \sum_{j=1}^{r_2-1} m_{ij} \ell_{ij},$$

for some $m = (m_{ij}) \in M(r_1 - 1, r_2 - 1; \mathbb{Z}_{\geq 0}) = M(k, n; \mathbb{Z}_{\geq 0})$. This m is uniquely determined. If some entries of u are negative integers, then $\prod p_{ij}^{u_{ij}} / \prod \Gamma(u_{ij} + 1) = 0$, because of $1/\Gamma(N) = 0$ for $N \in \mathbb{Z}_{\leq 0}$. We thus have

$$Z(\beta; p) = \sum_u \frac{p^u}{\prod_{a,b} \Gamma(u_{ab} + 1)} = \sum_{\substack{u=u_0 + \sum_i \sum_j m_{ij} \ell_{ij} \\ m \in M(k, n; \mathbb{Z})}} \frac{p^u}{\prod_{a,b} \Gamma(u_{ab} + 1)}.$$

If $u = (u_{ab}) = u_0 + \sum_i \sum_j m_{ij} \ell_{ij}$, then

$$p^u = p^{u_0} \cdot \prod_{i=1}^k \prod_{j=1}^n x_{ij}^{m_{ij}} = \prod_{i=1}^k p_{i1}^{-\alpha_i} \cdot \prod_{j=1}^n p_{r_1, j+1}^{\alpha_{k+j}} \cdot p_{r_1 1}^{\sum_{i=1}^k \alpha_i + \alpha_{k+n+1}} \cdot \prod_{i=1}^k \prod_{j=1}^n x_{ij}^{m_{ij}},$$

$$\frac{1}{\prod_{a=1}^{r_1} \prod_{b=1}^{r_2} \Gamma(u_{ab} + 1)} = \frac{1}{\Gamma_m(\alpha)}.$$

Hence, the proposition is proved. \square

Hereafter, we put α , x , and u_0 as in Proposition 7.1 and its proof. Then the normalizing constant is expressed as $Z(\beta; p) = p^{u_0} \cdot S(\alpha; x)$.

According to [TKT], the expectation $E[U_{ij}]$ of the (i, j) -cell is given as

$$p_{ij} \frac{\partial}{\partial p_{ij}} \log Z = \frac{1}{Z} \cdot p_{ij} \frac{\partial Z}{\partial p_{ij}}.$$

It is known that the expectations are functions in x (see also Corollary 7.2). Further, the values $\partial E[U_{ij}]/\partial x_{i'j'}$ are important to solve the conditional maximal likelihood estimate problem. We express the expectations and their derivatives by entries of $\mathbf{S}(\alpha; x)$ and $d_x \mathbf{S}(\alpha; x)$. Recall that $\mathbf{S}(\alpha; x)$ is aligned by the elements of $\hat{\mathcal{J}} = {}_{k+n+1}\mathcal{J}_0$. For $J \in \hat{\mathcal{J}}$, we call the entry of $\mathbf{S}(\alpha; x)$ corresponding to J by the J -entry.

Corollary 7.2. *For $1 \leq i, i' \leq k$ and $1 \leq j, j' \leq n$, let S_{ij} be the ${}_i\dot{J}_{k+j}$ -entry of $\mathbf{S}(\alpha; x)$, and let $S_{(ij)(i'j')}$ be the ${}_i\dot{J}_{k+j}$ -entry of the coefficient of $dx_{i'j'}$ in $d_x \mathbf{S}(\alpha; x) = \Psi(\alpha; x) \mathbf{S}(\alpha; x)$. We denote the (i, j) -entry of u_0 by $(u_0)_{ij}$. Then we have*

$$E[U_{ij}] = (u_0)_{ij} + \frac{(-1)^k}{S} \sum_{a=1}^k \sum_{b=1}^n (-1)^a (\ell_{ab})_{ij} \alpha_{k+b} S_{ab},$$

$$\frac{\partial E[U_{ij}]}{\partial x_{i'j'}} = \frac{(-1)^k}{S^2} \sum_{a=1}^k \sum_{b=1}^n (-1)^a (\ell_{ab})_{ij} \alpha_{k+b} \left(S \cdot S_{(ab)(i'j')} - (-1)^{k-i'} \frac{\alpha_{k+j'}}{x_{i'j'}} \cdot S_{ab} \cdot S_{i'j'} \right),$$

where

$$(\ell_{ab})_{ij} = \begin{cases} 1 & \text{if } (i, j) = (a, b+1) \text{ or } (r_1, 1), \\ -1 & \text{if } (i, j) = (a, 1) \text{ or } (r_1, b+1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As mentioned in Section 4, the $i\dot{J}_{k+j}$ -entry of $\mathbf{S}(\alpha; x)$ is

$$\begin{aligned} S_{ij} &= \text{sgn} \begin{pmatrix} 0 & \cdots & i-1 & i+1 & i+2 & \cdots & k & k+j \\ 0 & \cdots & i-1 & k+j & i+1 & \cdots & k-1 & k \end{pmatrix} \cdot \frac{|\tilde{x}\langle_i \dot{J}_{k+j}\rangle|}{\alpha_{k+j}} \frac{\partial S(\alpha; x)}{\partial x_{ij}} \\ &= (-1)^{k-i} \cdot \frac{x_{ij}}{\alpha_{k+j}} \frac{\partial S(\alpha; x)}{\partial x_{ij}}. \end{aligned}$$

Since $x_{ab} = p^{\ell_{ab}} = p_{a,b+1} p_{r_1,1} p_{a,1}^{-1} p_{r_1,b+1}^{-1}$, we obtain

$$\begin{aligned} E[U_{ij}] &= \frac{1}{Z} \cdot p_{ij} \frac{\partial Z}{\partial p_{ij}} = \frac{1}{p^{u_0} S(\alpha; x)} \cdot p_{ij} \frac{\partial}{\partial p_{ij}} (p^{u_0} S(\alpha; x)) \\ &= (u_0)_{ij} + \frac{1}{S} \sum_{a=1}^k \sum_{b=1}^n p_{ij} \frac{\partial x_{ab}}{\partial p_{ij}} \frac{\partial S}{\partial x_{ab}} \\ &= (u_0)_{ij} + \frac{1}{S} \sum_{a=1}^k \sum_{b=1}^n (\ell_{ab})_{ij} \cdot x_{ab} \frac{\partial S}{\partial x_{ab}} \\ &= (u_0)_{ij} + \frac{1}{S} \sum_{a=1}^k \sum_{b=1}^n (\ell_{ab})_{ij} \cdot (-1)^{k-a} \alpha_{k+b} S_{ab}. \end{aligned}$$

We can easily obtain the second equality from the first one by using $S_{(ij)(i'j')} = \partial S_{ij} / \partial x_{i'j'}$. \square

Remark 7.3. By using this corollary and the chain rule, we can also obtain the gradient $(\partial Z / \partial p_{ij})_{i,j}$ and the Hessian $(\partial^2 Z / \partial p_{ij} \partial p_{i'j'})_{(i,j),(i',j')}$ from $\mathbf{S}(\alpha; x)$ and $d_x \mathbf{S}(\alpha; x)$.

By Proposition 7.1 and Corollary 7.2, we can reduce the numerical evaluation of the normalizing constant, the expectations and their derivatives to that of $\mathbf{S}(\alpha; x)$ and $d_x \mathbf{S}(\alpha; x)$. To apply our results, we use the following lemma.

Lemma 7.4. *For any $\check{\alpha}^0 = (\alpha_1^0, \dots, \alpha_{k+n+1}^0) \in \mathbb{C}^{k+n+1}$, the series $S(\alpha; x)$ ($\alpha_0 = -\sum_{j=1}^{k+n+1} \alpha_j$) as a function in $(\alpha_1, \dots, \alpha_{k+n+1}; x) \in \mathbb{C}^{k+n+1} \times \mathbb{C}^{kn}$ converges uniformly on a small neighborhood of $(\check{\alpha}^0; O)$.*

Proof. This lemma can be shown in a similar way to [G, Lemma 7.3]. \square

We put $\alpha^0 := (-\sum_{j=1}^{k+n+1} \alpha_j^0, \alpha_1^0, \dots, \alpha_{k+n+1}^0)$. If $\alpha_i^0 \in \mathbb{Z}_{<0}$ ($1 \leq i \leq k$) and $\alpha_{k+j}^0 \in \mathbb{Z}_{>0}$ ($1 \leq j \leq n$), then the series $S(\alpha^0; x)$ becomes a polynomial in x_{ij} ($1 \leq i \leq k$, $1 \leq j \leq n$). This lemma implies that $\lim_{\alpha \rightarrow \alpha^0} S(\alpha; x)$ coincides with the polynomial $S(\alpha^0; x)$, for x in a small neighborhood of the zero matrix O . In a similar way, we can show that the partial derivatives of $S(\alpha; x^0)$ converge to those of the polynomial $S(\alpha^0; x)$, as $\alpha \rightarrow \alpha^0$. By the identity theorem for holomorphic functions, we obtain the following corollary.

Corollary 7.5. *Let α be the integer vector defined in Proposition 7.1. Then the relations in Corollaries 6.2 and 6.3 hold, as those between the vectors consisting of polynomials in x .*

By Proposition A.1 in Appendix A, we obtain the following lemma.

Lemma 7.6. *If none of α_i ($0 \leq i \leq k+n+1$) and $|\tilde{x}\langle J \rangle|$ ($J \in \mathcal{J}$) is zero, then the matrices $\Psi(\alpha; x)$ and $U_i(\alpha; x)$ are well-defined. Further, $U_i(\alpha; x)$ is invertible.*

We explain an algorithm to evaluate the normalizing constant by using the contiguity relations. We put

$$\alpha^0 := (1 - r_2, \underbrace{-1, \dots, -1}_{r_1-1}, \underbrace{1, \dots, 1}_{r_2-1}, r_1 - 1),$$

$$\delta_i := (-1, \underset{0}{0}, \underset{1}{0}, \dots, \underset{i-1}{0}, \underset{i}{1}, \underset{i+1}{0}, \dots, \underset{r_1+r_2-1}{0}), \quad i = 1, \dots, r_1 + r_2 - 1.$$

Algorithm 7.7.

Input: a parameter vector $\alpha = (\alpha_0, \dots, \alpha_{r_1+r_2-1})$ with

$$\alpha_i \in \mathbb{Z}_{<0} \quad (0 \leq i \leq r_1 - 1), \quad \alpha_{r_1+j} \in \mathbb{Z}_{>0} \quad (0 \leq j \leq r_2 - 1).$$

Output: a sequence $\{\alpha^l\}_{l=1}^e$ satisfying

- (i) $\alpha^e = \alpha$,
- (ii) each entry of $\alpha^l = (\alpha_0^l, \dots, \alpha_{r_1+r_2-1}^l)$ is nonzero,
- (iii) for $1 \leq l \leq e$, the difference $\alpha^l - \alpha^{l-1}$ is one of the following:

$$-\delta_1, \dots, -\delta_{r_1-1}, \delta_{r_1}, \dots, \delta_{r_1+r_2-2}, \pm \delta_{r_1+r_2-1}.$$

1. Let $\sigma = (\sigma_0, \dots, \sigma_{r_1+r_2-1}) := \alpha - \alpha^0$. Then $\sigma_1, \dots, \sigma_{r_1-1} \in \mathbb{Z}_{\leq 0}$ and $\sigma_{r_1}, \dots, \sigma_{r_1+r_2-2} \in \mathbb{Z}_{\geq 0}$.
2. Let $l := 0$. For j from 0 to $r_2 - 2$, while $\alpha_{r_1+j}^l < \alpha_{r_1+j}$, do

$$\alpha^{l+1} \leftarrow \alpha^l + \delta_{r_1+j},$$

$$l \leftarrow l + 1.$$

3. If $r_1 - 1 < \alpha_{r_1+r_2-1}$, while $\alpha_{r_1+r_2-1}^l < \alpha_{r_1+r_2-1}$, do

$$\begin{aligned}\alpha^{l+1} &\leftarrow \alpha^l + \delta_{r_1+r_2-1}, \\ l &\leftarrow l + 1.\end{aligned}$$

Else if $r_1 - 1 > \alpha_{r_1+r_2-1}$, while $\alpha_{r_1+r_2-1}^l > \alpha_{r_1+r_2-1}$, do

$$\begin{aligned}\alpha^{l+1} &\leftarrow \alpha^l - \delta_{r_1+r_2-1}, \\ l &\leftarrow l + 1.\end{aligned}$$

4. For i from 1 to $r_1 - 1$, while $\alpha_i^l > \alpha_i$, do

$$\begin{aligned}\alpha^{l+1} &\leftarrow \alpha^l - \delta_i, \\ l &\leftarrow l + 1.\end{aligned}$$

5. Return $\alpha^1, \dots, \alpha^{l-1}(=:\alpha^e)$.

By (ii) and Lemma 7.6, the matrix $U_i(\alpha^l; x)$ is well-defined and invertible for $1 \leq i \leq r_1 + r_2 - 1$ and $0 \leq l \leq e$.

Algorithm 7.8.

Input: marginal sums β and probabilities p .

Output: the normalizing constant $Z(\beta; p)$, the expectations $E[U_{ij}]$, and their derivatives $\partial E[U_{ij}]/\partial x_{i'j'}$.

1. Let α and x be as in Proposition 7.1.
2. By using Algorithm 7.7, find a sequence $\{\alpha^l\}_{l=1}^e$ satisfying (i), (ii), and (iii).
3. Compute $\mathbf{S}(\alpha^0; x)$ by the definition.
4. For l from 1 to e , evaluate $\mathbf{S}(\alpha^l; x)$ from $\mathbf{S}(\alpha^{l-1}; x)$, by multiplying $U_i^{\pm 1}$ as Corollary 6.3.
5. By Proposition 7.1 and Corollary 7.2, we obtain the numerical values of Z and $E[U_{ij}]$.
6. By the expressions (3.9) and (3.10), evaluate $\Psi(\alpha; x)$ and $\Psi(\alpha; x)\mathbf{S}(\alpha; x)(=d_x\mathbf{S}(\alpha; x))$.
7. By Corollary 7.2, we obtain the numerical values of $\partial E[U_{ij}]/\partial x_{i'j'}$.

Remark 7.9. Though the evaluation of $U_i^{\pm 1}$ needs the inverse matrices of intersection matrices, these have explicit expression; see Remark A.2 (1). Thus, it is not hard to evaluate $U_i^{\pm 1}$.

Example 7.10 ($r_1 = r_2 = 3$ ($k = n = 2$)). We consider 3×3 contingency tables whose marginal sums and probabilities are given as

follows, respectively.

$$\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \begin{array}{l} 2 \\ 3 \\ 3 \\ 8 \end{array} \quad p = \begin{array}{|c|c|c|} \hline 1 & 1/2 & 1/3 \\ \hline 1 & 1/5 & 1/7 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

In this case, the notations appearing in Algorithms 7.7 and 7.8 are as follows:

$$\alpha = (-3, -2, -3, 3, 4, 1), \quad x_{11} = \frac{1}{2}, \quad x_{12} = \frac{1}{3}, \quad x_{21} = \frac{1}{5}, \quad x_{22} = \frac{1}{7},$$

$$\alpha^0 = (-2, -1, -1, 1, 1, 2), \quad \sigma = (-1, -1, -2, 2, 3, -1), \quad e = 9.$$

We write down the changes of parameters (see Figure 1). Note that the G-M vector $\mathbf{S}(\alpha; x)$ is

$$\begin{aligned} \mathbf{S}(\alpha; x) &= (\text{constant}) \cdot \int_{\Delta\langle 125 \rangle} \prod_{j=1}^5 L_j^{\alpha_j} \cdot {}^t(\varphi\langle 012 \rangle, \varphi\langle 013 \rangle, \varphi\langle 014 \rangle, \varphi\langle 023 \rangle, \varphi\langle 024 \rangle, \varphi\langle 034 \rangle) \\ &= {}^t\left(S(\alpha; x), \frac{x_{21}}{\alpha_3} \cdot \frac{\partial S(\alpha; x)}{\partial x_{21}}, \frac{x_{22}}{\alpha_4} \cdot \frac{\partial S(\alpha; x)}{\partial x_{22}}, \right. \\ &\quad \left. \frac{-x_{11}}{\alpha_3} \cdot \frac{\partial S(\alpha; x)}{\partial x_{11}}, \frac{-x_{12}}{\alpha_4} \cdot \frac{\partial S(\alpha; x)}{\partial x_{12}}, \frac{x_{11}x_{22} - x_{12}x_{21}}{\alpha_3\alpha_4} \cdot \frac{\partial^2 S(\alpha; x)}{\partial x_{11}\partial x_{22}}\right). \end{aligned}$$

Example 7.11 ($r_1 = r_2 = 5$ ($k = n = 4$)). We consider the following case.

$$\begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} \begin{array}{l} 400 \\ 410 \\ 711 \\ 250 \\ 250 \\ 2021 \end{array}$$

$$\begin{array}{l} 560 \quad 361 \quad 550 \quad 350 \quad 200 \end{array}$$

Evaluation of the expectations takes 19003 seconds on our implementation (with Risa/Asir on a machine with an Intel Xeon (2.70GHz) and 256G memory).

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APPENDIX A. INVERSE OF INTERSECTION MATRICES

We regard α_i as an indeterminant, and entries of matrices as elements in the rational function field $\mathbb{C}(\alpha) = \mathbb{C}(\alpha_0, \dots, \alpha_{k+n+1})$ with a relation

$$\begin{aligned}
& \mathbf{S}(-2, -1, -1, 1, 1, 2; x) = \mathbf{S}(\alpha^0; x), \\
& \frac{1}{2}U_3(-2, -1, 1, 1, 1, 2; x) \times \downarrow \\
& \mathbf{S}(-3, -1, -1, 2, 1, 2; x) = \mathbf{S}(\alpha^1; x), \quad \alpha^1 = \alpha^0 + \delta_3, \\
& \frac{1}{3}U_3(-3, -1, -1, 2, 1, 2; x) \times \downarrow \\
& \mathbf{S}(-4, -1, -1, 3, 1, 2; x) = \mathbf{S}(\alpha^2; x), \quad \alpha^2 = \alpha^1 + \delta_3, \\
& \frac{1}{2}U_4(-4, -1, -1, 3, 1, 2; x) \times \downarrow \\
& \mathbf{S}(-5, -1, -1, 3, 2, 2; x) = \mathbf{S}(\alpha^3; x), \quad \alpha^3 = \alpha^2 + \delta_4, \\
& \frac{1}{3}U_4(-5, -1, -1, 3, 2, 2; x) \times \downarrow \\
& \mathbf{S}(-6, -1, -1, 3, 3, 2; x) = \mathbf{S}(\alpha^4; x), \quad \alpha^4 = \alpha^3 + \delta_4, \\
& \frac{1}{4}U_4(-6, -1, -1, 3, 3, 2; x) \times \downarrow \\
& \mathbf{S}(-7, -1, -1, 3, 4, 2; x) = \mathbf{S}(\alpha^5; x), \quad \alpha^5 = \alpha^4 + \delta_4, \\
& 2U_5^{-1}(-6, -1, -1, 3, 4, 1; x) \times \downarrow \\
& \mathbf{S}(-6, -1, -1, 3, 4, 1; x) = \mathbf{S}(\alpha^6; x), \quad \alpha^6 = \alpha^5 - \delta_5, \\
& U_1^{-1}(-5, -2, -1, 3, 4, 1; x) \times \downarrow \\
& \mathbf{S}(-5, -2, -1, 3, 4, 1; x) = \mathbf{S}(\alpha^7; x), \quad \alpha^7 = \alpha^6 - \delta_1, \\
& U_2^{-1}(-4, -2, -2, 3, 4, 1; x) \times \downarrow \\
& \mathbf{S}(-4, -2, -2, 3, 4, 1; x) = \mathbf{S}(\alpha^8; x), \quad \alpha^8 = \alpha^7 - \delta_2, \\
& U_2^{-1}(-3, -2, -3, 3, 4, 1; x) \times \downarrow \\
& \mathbf{S}(-3, -2, -3, 3, 4, 1; x) = \mathbf{S}(\alpha^9; x) = \mathbf{S}(\alpha; x), \quad \alpha^9 = \alpha^8 - \delta_2.
\end{aligned}$$

FIGURE 1. Step 4 in Algorithm 7.8.

$\sum_{i=0}^{k+n+1} \alpha_i = 0$. For $f(\alpha) \in \mathbb{C}(\alpha)$, we denote $f(-\alpha)$ by $f(\alpha)^\vee$. For a matrix $A \in M(r, r; \mathbb{C}(\alpha))$ ($r = \binom{k+n}{k}$), let A^\vee be the matrix operated $^\vee$ on each entry of A . In this appendix, we show the following proposition.

Proposition A.1. *For $p_1 \neq q_1$ and $p_2 \neq q_2$, we put*

$$C_{(p_1 q_1)(p_2 q_2)}(\alpha) := \left(\mathcal{I}(\varphi\langle I \rangle, \varphi\langle J \rangle) \right)_{I \in q_1 \mathcal{J}_{p_1}, J \in q_2 \mathcal{J}_{p_2}}$$

whose entries are regarded as rational functions of α_i 's. If none of $a_i \in \mathbb{C}(0 \leq i \leq k+n+1)$ is zero, then $C_{(p_1 q_1)(p_2 q_2)}(a_0, \dots, a_{k+n+1})$ is well-defined and invertible.

Proof. The well-definedness is clear by Fact 3.2. We show that the matrix is invertible. For $p \neq q$, there exists an invertible matrix $A_{pq} \in M(r, r; \mathbb{C}(\alpha))$ such that

$${}^t(\dots, \varphi\langle J \rangle, \dots)_{J \in {}_q \mathcal{J}_p} = A_{pq} {}^t(\dots, \varphi\langle I \rangle, \dots)_{I \in {}_{k+n+1} \mathcal{J}_0}.$$

Let $\{I_1, \dots, I_r\}$ (resp. $\{J_1, \dots, J_r\}$) be the set of subsets of $\{0, 1, \dots, k+n+1\} - \{p_1, q_1\}$ (resp. $\{0, 1, \dots, k+n+1\} - \{p_2, q_2\}$) with cardinality k . By Fact 3.2, we have

$$C_{(p_1 q_1)(q_1 p_1)} = (2\pi\sqrt{-1})^k \cdot \text{diag} \left(\frac{1}{\prod_{i \in I_1} \alpha_i}, \dots, \frac{1}{\prod_{i \in I_r} \alpha_j} \right),$$

$$C_{(q_2 p_2)(p_2 q_2)} = (2\pi\sqrt{-1})^k \cdot \text{diag} \left(\frac{1}{\prod_{j \in J_1} \alpha_j}, \dots, \frac{1}{\prod_{j \in J_r} \alpha_j} \right).$$

$\mathbb{C}(\alpha)$ -linearity of the intersection form \mathcal{I} leads

$$C_{(p_1 q_1)(p_2 q_2)} = A_{p_1 q_1} C {}^t A_{p_2 q_2}^\vee,$$

where $C := C_{(0, k+n+1)(0, k+n+1)}$. We thus have

$$\begin{aligned} (C_{(p_1 q_1)(p_2 q_2)})^{-1} &= ({}^t A_{p_2 q_2}^\vee)^{-1} C^{-1} A_{p_1 q_1}^{-1} = C_{(q_2 p_2)(p_2 q_2)}^{-1} A_{q_2 p_2} A_{p_1 q_1}^{-1} \\ &= C_{(q_2 p_2)(p_2 q_2)}^{-1} A_{q_2 p_2} C {}^t A_{q_1 p_1}^\vee C_{(p_1 q_1)(q_1 p_1)}^{-1} \\ &= C_{(q_2 p_2)(p_2 q_2)}^{-1} C_{(q_2 p_2)(q_1 p_1)} C_{(p_1 q_1)(q_1 p_1)}^{-1}. \end{aligned}$$

This equality holds in $M(r, r; \mathbb{C}(\alpha))$. If none of $a_i \in \mathbb{C}(0 \leq i \leq k+n+1)$ is zero, then $C_{(p_1 q_1)(p_2 q_2)}(a_0, \dots, a_{k+n+1}) \in M(r, r; \mathbb{C})$ is invertible, since the right-hand side is well-defined. \square

- Remark A.2.** (1) This proof gives an explicit expression of the inverse matrix of $C_{(p_1 q_1)(p_2 q_2)}$. It is written as a product of the intersection matrix and diagonal ones.
- (2) The matrices $C(\alpha)$, $P_i(\alpha)$, and $Q_i(\alpha)$ in Theorem 5.3 coincide with $C_{(0, k+n+1)(0, k+n+1)}(\alpha)$, $C_{(0, i)(0, k+n+1)}(\alpha)$, and $C_{(i, 0)(0, k+n+1)}(\alpha)$, respectively.

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